
Epidemic Model of HIV/AIDS Transmission Dynamics with Different Latent Stages Based on Treatment

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Abstract: The mathematical model for analyzing the transmission dynamics of HIV/AIDS epidemic with treatment is studied by considering the three latent compartments for slow, medium and fast progresses of developing the AIDS. By constructing the system of differential equations for the different population groups namely susceptible, three types of latent individuals, symptomatic stage group and full blown AIDS individuals, the mathematical analysis is carried out in order to understand the dynamics of disease spread. By determining the basic reproduction number (R_0), the model examines the two equilibrium points (i) the disease free equilibrium and (ii) the endemic equilibrium. It is established that if $R_0 < 1$, the disease free equilibrium is locally and globally asymptotically stable. The stability of endemic equilibrium has also been discussed.

Keywords: Transmission Dynamic, HIV/AIDS, Latent Compartments, Reproduction Number, Stability

1. Introduction

HIV/AIDS is one of the fatal diseases, which causes millions of death in both developed and developing nations. More than 35 million people approximately in the worldwide are living with HIV. In 2013, the newly infected persons were reported around 2.1 million which is 38% less from 2001. HIV infections among children are also declined by 58% since 2001. AIDS related deaths have also fallen by 35% since the peak in 2005, which is a result of availability of Antiretroviral Therapy (ART). Still tuberculosis related deaths in people living with HIV remain the leading cause of death. Since epidemic modeling initially studied by May and Anderson [1], many advance researches have taken place and several important aspects were incorporated by different authors during last three decades. Several epidemiological investigations related to HIV/AIDS infection have been conducted via mathematical models by many researchers [2-6]. An epidemiological model with nonlinear incident rate was discussed by several authors [7, 8]. Cai *et al.* [9] established the ordinary differential equation (ODE) model

with two infective stages before transition to AIDS. They considered by all sort of treatment methods by considering that some individuals transformed into asymptomatic individuals from symptomatic individuals. The capacity of human immune system can be reduced by chronic diseases, such as diabetes and tuberculosis and can reduce. To analyze this behavior, Huo and Feng [10] developed a model with slow and fast latent compartments. Okosun *et al.* [11] studied the treatment of HIV/AIDS and screening of unaware infective on the transmission dynamics of disease in a homogeneous population. Defeng and Wang [12] proposed a time delayed mathematical model to analyze the effect of vaccination and ART (Antiretroviral Therapy) on HIV/AIDS. They considered two types of individuals one who are aware of their infected stage and other individuals are unaware about their infected stage. Bhunu and Mushayabasa [13] formulated a mathematical model of the co-dynamics of hepatitis C virus and HIV/AIDS in order to assess their impact on the dynamics of each disease in the presence of treatment. Cai *et al.* [14] investigated an HIV/AIDS treatment model with multiple infection stages and treatment where infection was assumed to be of density dependent

form. Kaur *et al.* [15] also proposed a nonlinear model for studying the transmission dynamics of HIV/AIDS epidemic with emphasis on the role of female sex workers. Elaiw and Almualllem [16] to investigate the qualitative behaviors of three HIV dynamical models with two types of co circulating target cells. Wang *et al.* [17] has been studied the global stability of HIV viral infection model with continuous age-structure using the direct lyapunov method. Shen *et al.* [18] to analyze the mathematical model of global dynamics with two lyapunov functions are constructed to prove the global stability of disease free and endemic equilibria. Treatment class of HIV/AIDS epidemic model is introduced by Huo *et al.* [19].

In this paper, the mathematical model have been proposed for analyzing the transmission dynamics of HIV/AIDS epidemic with treatment by considering the three latent compartments for slow, medium and fast progresses of developing the AIDS. To study the dynamics of the spread of HIV, the basic reproduction number under disease free equilibrium is analyzed. The global stability of the endemic equilibrium for some special cases is discussed. The remaining contents of the paper are organized into different sections as follows. Section 2 presents the model description, system equations and some basic properties. Stability analysis is done in the section 3. Numerical results to support the proposed model are provided in section 4. Section 5 summaries the important findings and scope of the future works.

2. Model Description

The mathematical model to analyze the transmission dynamic of HIV/AIDS epidemic is developed by dividing the total population into six compartments, namely the

susceptible compartment (S), slow latent compartment (I_1), medium compartment (I_2), fast latent compartment (I_3), symptomatic stage (J) and a full-blown AIDS (A) group. The model flow depicting the biological system is illustrated in fig. 1. The total number of population at time t is given by

$$N(t) = S(t) + I_1(t) + I_2(t) + I_3(t) + J(t) + A(t).$$

For the mathematical formulation of the model, the following notations are used:

- Λ Recruitment rate of the population
- β_1 Transmission coefficient of the fast latent compartment
- β_2 Transmission coefficient of the symptomatic stage
- p_1, p_2 The fraction of susceptible S being infected by I_3 and entering into I_1, I_2 , respectively; $p_3 = 1 - p_1 - p_2$
- q_1, q_2 The fraction of susceptible S being infected by J and entering into I_1, I_2 , respectively; $q_3 = 1 - q_1 - q_2$
- ϵ_i Progression rate from latent compartment I_i to $I_{i+1}; i = 1, 2$
- r Progression rate from latent compartment I_3 to symptomatic compartment J
- ρ Progression rate of disease from compartment J to A
- ξ_i Treatment rate from compartment J to $I_i; i = 1, 2, 3$
- $\mu(\alpha)$ Natural (disease induced) death rate

2.1. The Governing Equations

The following differential equations governing the model are constructed by considering the appropriate in-flow and out-flow rates of each compartment:

$$\frac{dS(t)}{dt} = \Lambda - (p_1 + p_2 + p_3)\beta_1 I_3(t)S(t) - (q_1 + q_2 + q_3)\beta_2 J(t)S(t) - \mu S(t) \tag{1}$$

$$\frac{dI_1(t)}{dt} = p_1\beta_1 I_3(t)S(t) + q_1\beta_2 J(t)S(t) - (\epsilon_1 + \mu)I_1(t) + \xi_1 J(t) \tag{2}$$

$$\frac{dI_2(t)}{dt} = p_2\beta_1 I_3(t)S(t) + q_2\beta_2 J(t)S(t) + \epsilon_1 I_1(t) - (\epsilon_2 + \mu)I_2(t) + \xi_2 J(t) \tag{3}$$

$$\frac{dI_3(t)}{dt} = p_3\beta_1 I_3(t)S(t) + q_3\beta_2 J(t)S(t) + \epsilon_2 I_2(t) - (r + \mu)I_3(t) + \xi_3 J(t) \tag{4}$$

$$\frac{dJ(t)}{dt} = rI_3(t) - (\xi_1 + \xi_2 + \xi_3 + \rho + \mu)J(t) \tag{5}$$

$$\frac{dA(t)}{dt} = \rho J(t) - (\mu + \alpha)A(t) \tag{6}$$

For brevity of notation, by using

$$b_1 = \epsilon_1 + \mu, \quad b_2 = \epsilon_2 + \mu, \quad b_3 = r + \mu, \quad b_4 = \xi_1 + \xi_2 + \xi_3 + \rho + \mu, \text{ and } b_5 = \mu + \alpha$$

The above set of equations (1)-(6) becomes

$$\frac{dS(t)}{dt} = \Lambda - \beta_1 I_3(t)S(t) - \beta_2 J(t)S(t) - \mu S(t) \tag{7}$$

$$\frac{dI_1(t)}{dt} = p_1 \beta_1 I_3(t)S(t) + q_1 \beta_2 J(t)S(t) - b_1 I_1(t) + \xi_1 J(t) \tag{8}$$

$$\frac{dI_2(t)}{dt} = p_2 \beta_1 I_3(t)S(t) + q_2 \beta_2 J(t)S(t) + \varepsilon_1 I_1(t) - b_2 I_2(t) + \xi_2 J(t) \tag{9}$$

$$\frac{dI_3(t)}{dt} = p_3 \beta_1 I_3(t)S(t) + q_3 \beta_2 J(t)S(t) + \varepsilon_2 I_2(t) - b_3 I_3(t) + \xi_3 J(t) \tag{10}$$

$$\frac{dJ(t)}{dt} = r I_3(t) - b_4 J(t) \tag{11}$$

$$\frac{dA(t)}{dt} = \rho J(t) - b_5 A(t) \tag{12}$$

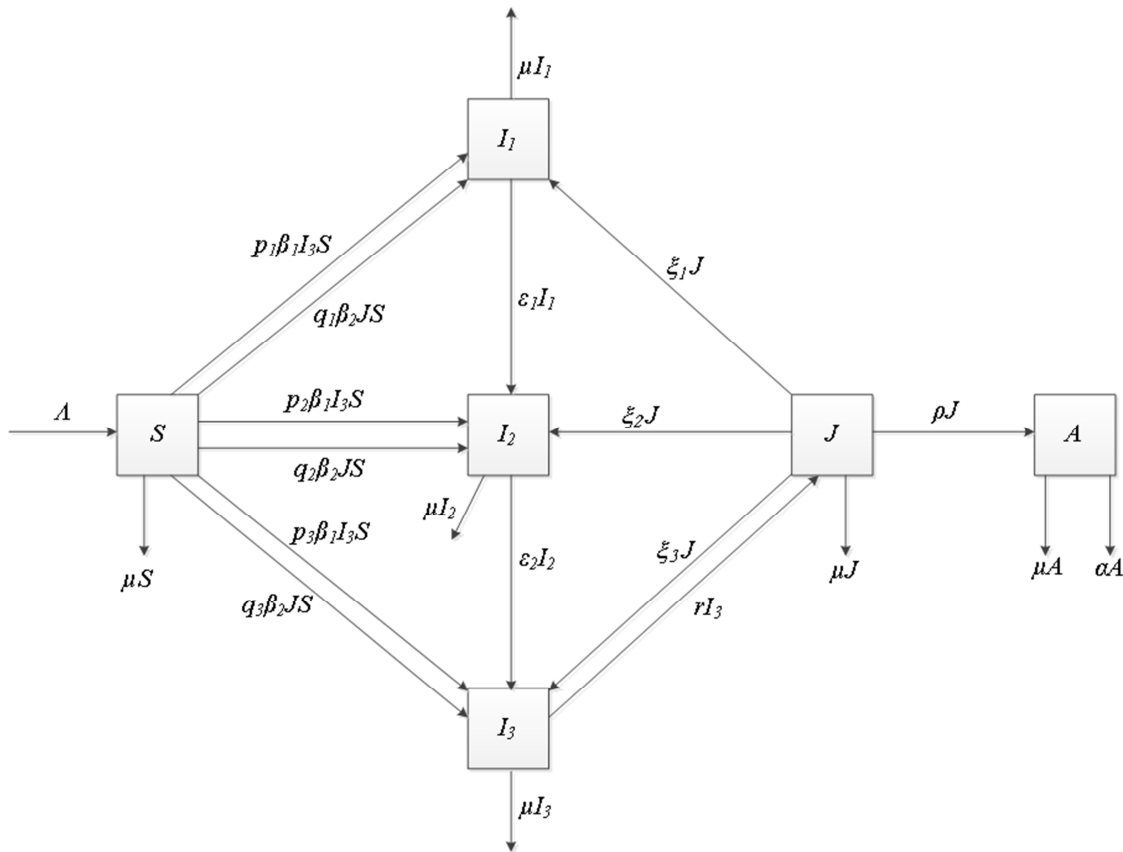


Fig. 1. Transmission dynamic diagram.

2.2. Basic Properties

In this sub-section, we discuss some preliminary concepts which will be needed further for the mathematical analysis of the concerned model. We will show that for $t \geq 0$, all the solutions are positively invariant in some region.

2.2.1. Invariant Region

Since the model displays the changes in the human population, the variables and the parameters are assumed to be positive for all $t \geq 0$. The system of equations (7)-(12) will therefore be analyzed in a suitable feasible region Ω of biological interest. We have the following lemma related to feasible region for the system (7)-(12).

Lemma1: The feasible region Ω defined by

$$\Omega = \left\{ (S(t), I_1(t), I_2(t), I_3(t), J(t), A(t)) \in R_+^6 : N(t) \leq \frac{\Lambda}{\mu} \right\}$$

with initial conditions

$$S(0) \geq 0, I_1(0) \geq 0, I_2(0) \geq 0, I_3(0) \geq 0, J(0) \geq 0, A(0) \geq 0$$

is positively invariant for the system of equations (7)-(12).

Proof: Adding the system of equations (1)-(6), we obtain

$$\frac{dN}{dt} = \Lambda - \mu N - \alpha A \leq \Lambda - \mu N. \tag{13}$$

Solving the above differential equation (13), we have

$$0 \leq N(t) \leq \frac{\Lambda}{\mu} + N(0)e^{-\mu t},$$

where $N(0)$ represents the initial values of the total population. Thus $\lim_{t \rightarrow +\infty} \sup N(t) \leq \frac{\Lambda}{\mu}$.

(i) there exists a first time t_1 such that

$$S(t_1) = 0, S'(t_1) < 0, I_1(t) \geq 0, I_2(t) \geq 0, I_3(t) \geq 0, J(t) \geq 0, A(t) \geq 0, 0 \leq t \leq t_1,$$

(ii) there exists a t_2

$$I_1(t_2) = 0, I_1'(t_2) < 0, S(t) \geq 0, I_2(t) \geq 0, I_3(t) \geq 0, J(t) \geq 0, A(t) \geq 0, 0 \leq t \leq t_2,$$

(iii) there exists a t_3

$$I_2(t_3) = 0, I_2'(t_3) < 0, S(t) \geq 0, I_1(t) \geq 0, I_3(t) \geq 0, J(t) \geq 0, A(t) \geq 0, 0 \leq t \leq t_3,$$

(iv) there exists a t_4

$$I_3(t_4) = 0, I_3'(t_4) < 0, S(t) \geq 0, I_1(t) \geq 0, I_2(t) \geq 0, J(t) \geq 0, A(t) \geq 0, 0 \leq t \leq t_4,$$

(v) there exists a t_5

$$J(t_5) = 0, J'(t_5) < 0, S(t) \geq 0, I_1(t) \geq 0, I_2(t) \geq 0, I_3(t) \geq 0, A(t) \geq 0, 0 \leq t \leq t_5,$$

(vi) there exists a t_6

$$A(t_6) = 0, A'(t_6) < 0, S(t) \geq 0, I_1(t) \geq 0, I_2(t) \geq 0, I_3(t) \geq 0, J(t) \geq 0, 0 \leq t \leq t_6,$$

In the first case (i), we have

$$S'(t_1) = \Lambda > 0,$$

which is a contradiction meaning that $S(t) \geq 0, t \geq 0$.

In the second case (ii), we have

$$I_1'(t_2) = p_1\beta_1 I_3(t_2)S(t_2) + q_1\beta_2 J(t_2)S(t_2) + \xi_1 J(t_2) \geq 0,$$

which is a contradiction meaning that $I_1(t) \geq 0, t \geq 0$.

Similarly, from (iii)–(vi), it can be easily shown that $I_2(0) > 0, I_3(0) > 0, J(0) > 0$, and $A(0) > 0$, for all $t \geq 0$. Thus, the

It implies that the region

$$\Omega = \left\{ (S(t), I_1(t), I_2(t), I_3(t), J(t), A(t)) \in R_+^6 : N(t) \leq \frac{\Lambda}{\mu} \right\}$$

is a positively invariant set for the system (7)-(12). Now we consider the dynamics of system (7)-(12) on the region given by set Ω .

2.2.2. Positivity of Solutions

It is important for the model described by the system (7)-(12) to prove that all the state variables are non-negative so that the solutions of the system with positive initial conditions remain positive for all $t > 0$. We thus state the following lemma.

Lemma 2: Given that the initial conditions of the system (7)-(12) by $S(0) > 0, I_1(0) > 0, I_2(0) > 0, I_3(0) > 0, J(0) > 0$, and $A(0) > 0$, the solutions $S(t), I_1(t), I_2(t), I_3(t), J(t)$ and $A(t)$ are non-negative for all $t > 0$.

Proof: Under the given initial conditions, it is easy to prove that the solutions of the system (7)-(12) are positive; if not, we assume a contradiction that:

solutions $S(t), I_1(t), I_2(t), I_3(t), J(t)$ and $A(t)$ of system (7)-(12) remain positive for all $t > 0$.

3. The Analysis

In this section, we compute the equilibrium states, namely the disease free equilibrium (DFE) and the endemic equilibrium (EE) and provide stability analysis by determining the basic reproduction number.

3.1. Disease Free Equilibrium (DFE) and Basic Reproduction Number R_0

Proposed model (7)-(12) has a disease free equilibrium given by

$$E_0 = (S^0, I_1^0, I_2^0, I_3^0, J^0, A^0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0 \right)$$

The basic reproduction number of the system (7)-(12) is obtained by considering the next generation matrices F and V defined for the appearance of new infection terms and the transfer of individuals out of latent compartment, respectively [20]. Let $X = (I_1, I_2, I_3, J, A, S)^T$, then the system (7)-(12) can be written as

$$\frac{dX}{dt} = F(X) - V(X) \tag{14}$$

where

$$F(X) = \begin{bmatrix} p_1\beta_1 I_3 S + q_1\beta_2 JS \\ p_2\beta_1 I_3 S + q_2\beta_2 JS \\ p_3\beta_1 I_3 S + q_3\beta_2 JS \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

and

$$V(X) = \begin{bmatrix} b_1 I_1 - \xi_1 J \\ b_2 I_2 - \xi_2 J - \varepsilon_1 I_1 \\ b_3 I_3 - \xi_3 J - \varepsilon_2 I_2 \\ b_4 J - r I_3 \\ b_5 A - \rho J \\ \beta_1 I_3 S + \beta_2 JS + \mu S - \Lambda \end{bmatrix} \tag{16}$$

Now we employ the linearization method. On taking partial derivatives, the associated matrices at DFE are obtained as,

$$D_i = \left[\frac{p_i\beta_i (b_1 b_2 \xi_3 + b_1 \varepsilon_2 \xi_2 + \varepsilon_1 \varepsilon_2 \xi_1) + q_i\beta_2 b_1 b_2 b_3}{W} \right] \frac{\Lambda}{\mu}; i = 1, 2, 3$$

$$W = b_1 b_2 b_3 b_4 - b_1 b_2 \xi_3 r - b_1 \varepsilon_2 \xi_2 r - \varepsilon_1 \varepsilon_2 \xi_1 r$$

Thus the reproduction number R_0 , is obtained as

$$R_0 = \rho(FV^{-1}) = \left[\frac{(p_1\beta_1 b_4 + q_1\beta_2 r)\varepsilon_1 \varepsilon_2 + (p_2\beta_1 b_4 + q_2\beta_2 r)b_1 \varepsilon_2 + (p_3\beta_1 b_4 + q_3\beta_2 r)b_1 b_2}{b_1 b_2 (b_3 b_4 - \xi_3 r) - \varepsilon_2 r (b_1 \xi_2 - \varepsilon_1 \xi_1)} \right] \frac{\Lambda}{\mu} \tag{20}$$

Following Theorem 2 of Van den Driessche and Watmough [17], we have the following result on the local stability of E_0 :
 Theorem 1: The disease free equilibrium E_0 of the system (7)-(12) is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

$$F = \begin{bmatrix} 0 & 0 & p_1\beta_1 \frac{\Lambda}{\mu} & q_1\beta_2 \frac{\Lambda}{\mu} & 0 \\ 0 & 0 & p_2\beta_1 \frac{\Lambda}{\mu} & q_2\beta_2 \frac{\Lambda}{\mu} & 0 \\ 0 & 0 & p_3\beta_1 \frac{\Lambda}{\mu} & q_3\beta_2 \frac{\Lambda}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

and

$$V = \begin{bmatrix} b_1 & 0 & 0 & -\xi_1 & 0 \\ -\varepsilon_1 & b_2 & 0 & -\xi_2 & 0 \\ 0 & -\varepsilon_2 & b_3 & -\xi_3 & 0 \\ 0 & 0 & -r & b_4 & 0 \\ 0 & 0 & 0 & -\rho & b_5 \end{bmatrix} \tag{18}$$

Consider the following matrix

$$FV^{-1} = \begin{bmatrix} A_1 & B_1 & C_1 & D_1 & 0 \\ A_2 & B_2 & C_2 & D_2 & 0 \\ A_3 & B_3 & C_3 & D_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

where

$$A_i = \left[\frac{p_i\beta_i b_4 \varepsilon_1 \varepsilon_2 + q_i\beta_2 \varepsilon_1 \varepsilon_2 r}{W} \right] \frac{\Lambda}{\mu}; i = 1, 2, 3$$

$$B_i = \left[\frac{p_i\beta_i b_1 b_4 \varepsilon_2 + q_i\beta_2 b_1 \varepsilon_2 r}{W} \right] \frac{\Lambda}{\mu}; i = 1, 2, 3$$

$$C_i = \left[\frac{p_i\beta_i b_1 b_2 b_4 + q_i\beta_2 b_1 b_2 r}{W} \right] \frac{\Lambda}{\mu}; i = 1, 2, 3$$

3.2. Global Stability of Disease Free Equilibrium (DFE)

Here we analyze the global stability by using a comparison theorem [21, 22].

Theorem 2: The disease free equilibrium E_0 of the system

(7)-(12) is globally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Proof: The equation of the infected components in system (7)-(12) can be written as

$$\begin{bmatrix} I_1' \\ I_2' \\ I_3' \\ J' \\ A' \end{bmatrix} = [F - V] \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ J \\ A \end{bmatrix} - \left[1 - \frac{\mu}{\Lambda} S \right] \begin{bmatrix} 0 & 0 & p_1\beta_1 \frac{\Lambda}{\mu} & q_1\beta_2 \frac{\Lambda}{\mu} & 0 \\ 0 & 0 & p_2\beta_1 \frac{\Lambda}{\mu} & q_2\beta_2 \frac{\Lambda}{\mu} & 0 \\ 0 & 0 & p_3\beta_1 \frac{\Lambda}{\mu} & q_3\beta_2 \frac{\Lambda}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ J \\ A \end{bmatrix} \tag{21}$$

where F and V are defined in (17) and (18) for all $t \geq 0$ in Ω .

Thus, we have

$$\begin{bmatrix} I_1' \\ I_2' \\ I_3' \\ J' \\ A' \end{bmatrix} \leq [F - V] \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ J \\ A \end{bmatrix} \tag{22}$$

Since all eigen values of the matrix $F - V$ have negative real parts, the linearized differential inequality (22) is stable whenever $R_0 < 1$. Consequently $(I_1, I_2, I_3, J, A) \rightarrow (0, 0, 0, 0, 0)$ as $t \rightarrow \infty$. Thus following the comparison theorem, we have $(I_1, I_2, I_3, J, A) \rightarrow (0, 0, 0, 0, 0)$

and $S \rightarrow \frac{\Lambda}{\mu}$ as $t \rightarrow \infty$.

Hence the DFE (E_0) is globally asymptotically stable for $R_0 < 1$.

3.3. Endemic Equilibrium (EE)

The endemic equilibrium point of the system (7)-(12) is given

$$V_1 = (S - S^* \ln S) + B(I_1 - I_1^* \ln I_1) + C(I_2 - I_2^* \ln I_2) + D(I_3 - I_3^* \ln I_3) + E(J - J^* \ln J) \tag{24}$$

The derivative of V_1 is

$$\begin{aligned} V_1' &= \left(1 - \frac{S^*}{S}\right) S' + B \left(1 - \frac{I_1^*}{I_1}\right) I_1' + C \left(1 - \frac{I_2^*}{I_2}\right) I_2' + D \left(1 - \frac{I_3^*}{I_3}\right) I_3' + E \left(1 - \frac{J^*}{J}\right) J' \\ &= \left(1 - \frac{S^*}{S}\right) [\Lambda - (\beta_1 I_3 S + \beta_2 JS) - \mu S] + B \left(1 - \frac{I_1^*}{I_1}\right) [p_1 \beta_1 I_3 S + q_1 \beta_2 JS + \xi_1 J - b_1 I_1] \\ &+ C \left(1 - \frac{I_2^*}{I_2}\right) [p_2 \beta_1 I_3 S + q_2 \beta_2 JS + \varepsilon_1 I_1 + \xi_2 J - b_2 I_2] \\ &+ D \left(1 - \frac{I_3^*}{I_3}\right) [p_3 \beta_1 I_3 S + q_3 \beta_2 JS + \varepsilon_2 I_2 + \xi_3 J - b_3 I_3] + E \left(1 - \frac{J^*}{J}\right) [r I_3 - b_4 J] \end{aligned} \tag{25}$$

$$E^* = (S^*, I_1^*, I_2^*, I_3^*, J^*, A^*); \quad S^* = \frac{\Lambda}{\mu R_0}$$

$$I_1^* = \frac{1}{b_1} \frac{\Lambda}{[\beta_1 b_4 + \beta_2 r] J^* + \mu r} [p_1 \beta_1 b_4 + r q_1 \beta_2] J^* + \frac{1}{b_1} \xi_1 J^*$$

$$I_2^* = -\frac{1}{\varepsilon_2} \frac{\Lambda}{[\beta_1 b_4 + \beta_2 r] J^* + \mu r} [p_3 \beta_1 b_4 + r q_1 \beta_2] J^* + \frac{[b_3 b_4 - \xi_3 r]}{\varepsilon_2 r} J^*$$

$$I_3^* = \frac{b_4}{r} J^*; \quad J^* = \frac{\mu r (R_0 - 1)}{[\beta_1 b_4 + \beta_2 r]}; \quad A^* = \frac{\rho}{b_5} J^* \tag{23}$$

3.4. Global Stability of Endemic Equilibrium

In this sub section, we present the global stability of endemic equilibrium.

Theorem 3: If $p_1 = q_1, p_2 = q_2, p_3 = q_3$ and $R_0 > 1$, the endemic equilibrium E^* is globally asymptotically stable.

Proof: To prove global stability of endemic equilibrium, we compute the following Lyapunov function:

The system (7)-(12) satisfies the following relation at the equilibrium point:

$$\Lambda = \beta_1 I_3^* S^* + \beta_2 J^* S^* + \mu S^* \tag{26}$$

$$b_1 = \frac{p_1 \beta_1 I_3^* S^* + q_1 \beta_2 J^* S^* + \xi_1 J^*}{I_1^*} \tag{27}$$

$$b_2 = \frac{p_2 \beta_1 I_3^* S^* + q_2 \beta_2 J^* S^* + \varepsilon_1 I_1^* + \xi_2 J^*}{I_2^*} \tag{28}$$

$$b_3 = \frac{p_3 \beta_1 I_3^* S^* + q_3 \beta_2 J^* S^* + \varepsilon_2 I_2^* + \xi_3 J^*}{I_3^*} \tag{29}$$

$$b_4 = \frac{r I_3^*}{J^*} \tag{30}$$

Substituting the results from (26)-(30) in (25), we get

$$\begin{aligned} V_1' = & \left(1 - \frac{S^*}{S}\right) \left[\beta_1 I_3^* S^* + \beta_2 J^* S^* + \mu S^* - (\beta_1 I_3 S + \beta_2 J S + \mu S) \right] \\ & + B \left(1 - \frac{I_1^*}{I_1}\right) \left[p_1 \beta_1 I_3 S + q_1 \beta_2 J S + \xi_1 J - \frac{p_1 \beta_1 I_3^* S^* + q_1 \beta_2 J^* S^* + \xi_1 J^*}{I_1^*} I_1 \right] \\ & + C \left(1 - \frac{I_2^*}{I_2}\right) \left[p_2 \beta_1 I_3 S + q_2 \beta_2 J S + \varepsilon_1 I_1 + \xi_2 J - \frac{p_2 \beta_1 I_3^* S^* + q_2 \beta_2 J^* S^* + \varepsilon_1 I_1^* + \xi_2 J^*}{I_2^*} I_2 \right] \\ & + D \left(1 - \frac{I_3^*}{I_3}\right) \left[p_3 \beta_1 I_3 S + q_3 \beta_2 J S + \varepsilon_2 I_2 + \xi_3 J - \frac{p_3 \beta_1 I_3^* S^* + q_3 \beta_2 J^* S^* + \varepsilon_2 I_2^* + \xi_3 J^*}{I_3^*} I_3 \right] \\ & + E \left(1 - \frac{J^*}{J}\right) \left[r I_3 - \frac{r I_3^*}{J^*} J \right] \end{aligned} \tag{31}$$

Using the following variables substitutions

$$\frac{S}{S^*} = x, \frac{I_1}{I_1^*} = y, \frac{I_2}{I_2^*} = z, \frac{I_3}{I_3^*} = u, \frac{J}{J^*} = v \tag{32}$$

Equation (31) reduces to

$$\begin{aligned} V_1' = & -\mu S^* \frac{(1-x)^2}{x} + \left(1 - \frac{1}{x}\right) \left[\beta_1 I_3^* S^* (1-xu) + \beta_2 J^* S^* (1-xv) \right] \\ & + B \left(1 - \frac{1}{y}\right) \left[p_1 \beta_1 I_3^* S^* (xu - y) + q_1 \beta_2 J^* S^* (xv - y) + \xi_1 J^* (v - y) \right] \\ & + C \left(1 - \frac{1}{z}\right) \left[p_2 \beta_1 I_3^* S^* (xu - z) + q_2 \beta_2 J^* S^* (xv - z) + \varepsilon_1 I_1^* (y - z) + \xi_2 J^* (v - z) \right] \\ & + D \left(1 - \frac{1}{u}\right) \left[p_3 \beta_1 I_3^* S^* (xu - u) + q_3 \beta_2 J^* S^* (xv - u) + \varepsilon_2 I_2^* (z - u) + \xi_3 J^* (v - u) \right] \\ & + E \left(1 - \frac{1}{v}\right) \left[r I_3^* (u - v) \right] \end{aligned}$$

$$\begin{aligned}
 &= -\mu S^* \frac{(1-x)^2}{x} + \beta_1 I_3^* S^* + \beta_2 J^* S^* + Bp_1 \beta_1 I_3^* S^* + Bq_1 \beta_2 J^* S^* + B\xi_1 J^* + Cp_2 \beta_1 I_3^* S^* \\
 &+ Cq_2 \beta_2 J^* S^* + C\varepsilon_1 I_1^* + C\xi_2 J^* + Dp_3 \beta_1 I_3^* S^* + Dq_3 \beta_2 J^* S^* + D\varepsilon_2 I_2^* + D\xi_3 J^* + ErI_3^* \\
 &+ xu[-\beta_1 I_3^* S^* + Bp_1 \beta_1 I_3^* S^* + Cp_2 \beta_1 I_3^* S^* + Dp_3 \beta_1 I_3^* S^*] \\
 &+ xv[-\beta_2 J^* S^* + Bq_1 \beta_2 J^* S^* + Cq_2 \beta_2 J^* S^* + Dq_3 \beta_2 J^* S^*] \\
 &+ u[\beta_1 I_3^* S^* - Dp_3 \beta_1 I_3^* S^* - Dq_3 \beta_2 J^* S^* - D\varepsilon_2 I_2^* - D\xi_3 J^* + ErI_3^*] \\
 &+ y[\beta_2 J^* S^* + B\xi_1 J^* + C\xi_2 J^* + D\xi_3 J^* - ErI_3^*] + y[-Bp_1 \beta_1 I_3^* S^* - Bq_1 \beta_2 J^* S^* - B\xi_1 J^* + C\varepsilon_1 I_1^*] \\
 &+ z[-Cp_2 \beta_1 I_3^* S^* - Cq_2 \beta_2 J^* S^* - C\varepsilon_1 I_1^* - C\xi_2 J^* + D\varepsilon_2 I_2^*] - \frac{xu}{y}(Bp_1 \beta_1 I_3^* S^*) - \frac{xv}{y}(Bq_1 \beta_2 J^* S^*) \\
 &- \frac{xu}{z}(Cp_2 \beta_1 I_3^* S^*) - \frac{xv}{z}(Cq_2 \beta_2 J^* S^*) - x(Dp_3 \beta_1 I_3^* S^*) - \frac{xv}{u}(Dq_3 \beta_2 J^* S^*) - \frac{v}{y}(B\xi_1 J^*) \\
 &- \frac{y}{z}(C\varepsilon_1 I_1^*) - \frac{v}{z}(C\xi_2 J^*) - \frac{z}{u}(D\varepsilon_2 I_2^*) - \frac{v}{u}(D\xi_3 J^*) - \frac{u}{v}(ErI_3^*) - \frac{1}{x}(\beta_1 I_3^* S^* + \beta_2 J^* S^*)
 \end{aligned} \tag{33}$$

The only variable terms that appears in (33) with positive coefficients are xu, xv, u, v, y and z . Making the coefficient of xu, xv, u, v, y and z equal to zero, we have

$$Bp_1 + Cp_2 + Dp_3 - 1 = 0 \tag{34}$$

$$Bq_1 + Cq_2 + Dq_3 - 1 = 0 \tag{35}$$

$$\beta_2 S^* + B\xi_1 + C\xi_2 + D\xi_3 - Eb_4 = 0 \tag{36}$$

$$\beta_1 S^* + ErI_3^* - Db_3 = 0 \tag{37}$$

$$C\varepsilon_1 - Bb_1 = 0 \tag{38}$$

$$D\varepsilon_2 - Cb_2 = 0 \tag{39}$$

Solving equations (34), (38) and (39), we get

$$B = \frac{\varepsilon_1 \varepsilon_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2}, C = \frac{b_1 \varepsilon_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2}, D = \frac{b_1 b_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} \tag{40}$$

From equations (36) and (37), we have

$$E = \frac{\beta_2 S^*}{b_4} + \frac{\varepsilon_1 \varepsilon_2 \xi_1 + \varepsilon_2 b_1 \xi_2 + b_1 b_2 \xi_3}{b_4 (p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2)} \text{ and } E = \frac{b_1 b_2 b_3}{r (p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2)} - \frac{\beta_1 S^*}{r} \tag{41}$$

Thus, we get

$$\begin{aligned}
 &\left[\frac{\beta_2 S^*}{b_4} + \frac{\varepsilon_1 \varepsilon_2 \xi_1 + \varepsilon_2 b_1 \xi_2 + b_1 b_2 \xi_3}{b_4 (p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2)} \right] - \left[\frac{b_1 b_2 b_3}{r (p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2)} - \frac{\beta_1 S^*}{r} \right] \\
 &= \frac{(\beta_1 b_4 + \beta_2 r) S^*}{b_4 r} - \frac{b_1 b_2 b_3 b_4 + (\varepsilon_1 \varepsilon_2 \xi_1 + \varepsilon_2 b_1 \xi_2 + b_1 b_2 \xi_3) r}{b_4 r (p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2)} \\
 &= \frac{(p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2)(\beta_1 b_4 + \beta_2 r) S^* - [b_1 b_2 b_3 b_4 + (\varepsilon_1 \varepsilon_2 \xi_1 + \varepsilon_2 b_1 \xi_2 + b_1 b_2 \xi_3) r]}{b_4 r (p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{(b_1 b_2 b_3 b_4 - b_1 b_2 \xi_3 r - b_1 \varepsilon_2 \xi_2 r - \varepsilon_1 \varepsilon_2 \xi_1 r) R_0 \cdot \frac{\mu}{\Lambda} \frac{\Lambda}{\mu R_0} - [b_1 b_2 b_3 b_4 + (\varepsilon_1 \varepsilon_2 \xi_1 + \varepsilon_2 b_1 \xi_2 + b_1 b_2 \xi_3) r]}{b_4 r (p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2)} \\
 &= 0
 \end{aligned}
 \tag{42}$$

Hence, equation (33) becomes

$$\begin{aligned}
 V_1' &= -\mu S^* \frac{(1-x)^2}{x} + \beta_1 I_3^* S^* + \beta_2 J^* S^* + B p_1 \beta_1 I_3^* S^* + B q_1 \beta_2 J^* S^* + B \xi_1 J^* + C p_2 \beta_1 I_3^* S^* \\
 &\quad + C q_2 \beta_2 J^* S^* + C \varepsilon_1 I_1^* + C \xi_2 J^* + D p_3 \beta_1 I_3^* S^* + D q_3 \beta_2 J^* S^* + D \varepsilon_2 I_2^* + D \xi_3 J^* \\
 &\quad + E r I_3^* - \frac{xu}{y} (B p_1 \beta_1 I_3^* S^*) - \frac{xv}{y} (B q_1 \beta_2 J^* S^*) - \frac{xu}{z} (C p_2 \beta_1 I_3^* S^*) - \frac{xv}{z} (C q_2 \beta_2 J^* S^*) \\
 &\quad - x (D p_3 \beta_1 I_3^* S^*) - \frac{xv}{u} (D q_3 \beta_2 J^* S^*) - \frac{y}{y} (B \xi_1 J^*) - \frac{y}{z} (C \varepsilon_1 I_1^*) - \frac{y}{z} (C \xi_2 J^*) - \frac{z}{u} (D \varepsilon_2 I_2^*) \\
 &\quad - \frac{v}{u} (D \xi_3 J^*) - \frac{u}{v} (E r I_3^*) - \frac{1}{x} (\beta_1 I_3^* S^* + \beta_2 J^* S^*)
 \end{aligned}
 \tag{43}$$

Thus, we have

$$\begin{aligned}
 V_1' &= -\mu S^* \frac{(1-x)^2}{x} + \frac{b_1 b_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} p_3 \beta_1 I_3^* S^* \left(2 - x - \frac{1}{x} \right) \\
 &\quad + \frac{b_1 b_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} \xi_3 J^* \left(2 - \frac{v}{u} - \frac{u}{v} \right) + \frac{b_1 \varepsilon_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} \xi_2 J^* \left(3 - \frac{v}{z} - \frac{z}{u} - \frac{u}{v} \right) \\
 &\quad + \frac{b_1 \varepsilon_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} p_2 \beta_1 I_3^* S^* \left(3 - \frac{1}{x} - \frac{z}{u} - \frac{xu}{z} \right) \\
 &\quad + \frac{b_1 b_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} q_3 \beta_2 J^* S^* \left(3 - \frac{1}{x} - \frac{u}{v} - \frac{xv}{u} \right) \\
 &\quad + \frac{\varepsilon_1 \varepsilon_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} p_1 \beta_1 I_3^* S^* \left(4 - \frac{1}{x} - \frac{z}{u} - \frac{y}{z} - \frac{xu}{y} \right) \\
 &\quad + \frac{\varepsilon_1 \varepsilon_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} q_1 \beta_2 J^* S^* \left(5 - \frac{1}{x} - \frac{z}{u} - \frac{u}{v} - \frac{y}{z} - \frac{xv}{y} \right) \\
 &\quad + \frac{b_1 \varepsilon_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} q_2 \beta_2 J^* S^* \left(4 - \frac{1}{x} - \frac{z}{u} - \frac{u}{v} - \frac{xv}{z} \right) \\
 &\quad + \frac{\varepsilon_1 \varepsilon_2}{p_1 \varepsilon_1 \varepsilon_2 + p_2 b_1 \varepsilon_2 + p_3 b_1 b_2} \xi_1 J^* \left(4 - \frac{v}{y} - \frac{y}{z} - \frac{u}{v} - \frac{z}{u} \right)
 \end{aligned}
 \tag{44}$$

Since the arithmetic mean is greater than or equal to the geometric mean, we have

- (i) $2 - x - \frac{1}{x} \leq 0$ for $x > 0$ and $2 - x - \frac{1}{x} = 0$ if and only if $x = 1$;
- (ii) $2 - \frac{v}{u} - \frac{u}{v} \leq 0$ for $u, v > 0$ and $2 - \frac{v}{u} - \frac{u}{v} = 0$ if and only if $u = v$;
- (iii) $3 - \frac{v}{z} - \frac{z}{u} - \frac{u}{v} \leq 0$ for $u, v, z > 0$ and $3 - \frac{v}{z} - \frac{z}{u} - \frac{u}{v} = 0$ if and only if $u = v = z$;
- (iv) $3 - \frac{1}{x} - \frac{z}{u} - \frac{xu}{z} \leq 0$ for $x, u, z > 0$ and $3 - \frac{1}{x} - \frac{z}{u} - \frac{xu}{z} = 0$ if and only if $x = 1, u = z$;

- (v) $3 - \frac{1}{x} - \frac{u}{v} - \frac{xv}{u} \leq 0$ for $x, u, v > 0$ and $3 - \frac{1}{x} - \frac{u}{v} - \frac{xv}{u} = 0$ if and only if $x = 1, u = v$;
- (vi) $4 - \frac{1}{x} - \frac{z}{u} - \frac{y}{z} - \frac{xu}{y} \leq 0$ for $x, y, z, u > 0$ and $4 - \frac{1}{x} - \frac{z}{u} - \frac{y}{z} - \frac{xu}{y} = 0$ if and only if $x = 1, y = z = u$;
- (vii) $5 - \frac{1}{x} - \frac{z}{u} - \frac{u}{v} - \frac{y}{z} - \frac{xv}{y} \leq 0$ for $x, y, z, u, v > 0$ and $5 - \frac{1}{x} - \frac{z}{u} - \frac{u}{v} - \frac{y}{z} - \frac{xv}{y} = 0$ if and only if $x = 1, y = z = u = v$;
- (viii) $4 - \frac{1}{x} - \frac{z}{u} - \frac{u}{v} - \frac{xv}{z} \leq 0$ for $x, z, u, v > 0$ and $4 - \frac{1}{x} - \frac{z}{u} - \frac{u}{v} - \frac{xv}{z} = 0$ if and only if $x = 1, u = v = z$;
- (ix) $4 - \frac{v}{y} - \frac{y}{z} - \frac{u}{v} - \frac{z}{u} \leq 0$ for $y, z, u, v > 0$ and $4 - \frac{v}{y} - \frac{y}{z} - \frac{u}{v} - \frac{z}{u} = 0$ if and only if $y = z = u = v$;

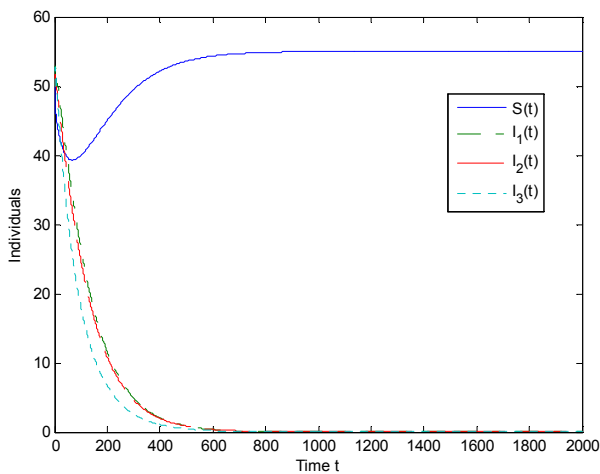
Therefore, $V_1' \leq 0$ for $x, y, z, u, v > 0$ and $V_1' = 0$ if and only if $x = 1, y = z = u = v$. The maximum invariant set of system (7)-(12) on the set $\{(x, y, z, u, v) : V_1' = 0\}$ is the singleton $(1, 1, 1, 1, 1)$. Thus for system (7)-(12), the endemic equilibrium E^* is globally asymptotically stable if $p_1 = q_1, p_2 = q_2, p_3 = q_3$, and $R_0 > 1$ by LaSalle [23].

4. Numerical Simulation

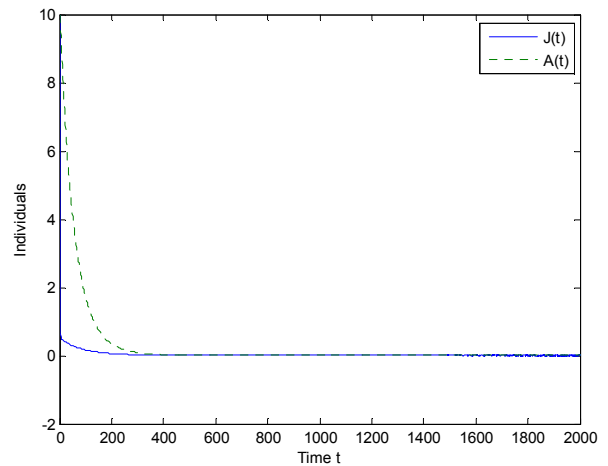
To validate the analytical results established in previous section, we conduct the numerical simulation by taking an example. The system (7)-(12) is simulated by fixing the default values of the parameters as

$$\Lambda = 0.55, \beta_1 = 0.0001, \beta_2 = 0.006, \epsilon_1 = 0.002, \epsilon_2 = 0.003, \\ r = 0.01, \rho = 0.03, \mu = 0.01, \alpha = 0.01.$$

For the other parameters chosen as $p_1 = 0.5, p_2 = 0.3, p_3 = 0.2, q_1 = 0.4, q_2 = 0.3, q_3 = 0.3, \xi_1 = 0.4, \xi_2 = 0.2$ and $\xi_3 = 0.4$, figs. 2(a)-2(b) demonstrate that the reproduction number $R_0 = 0.18041 < 1$ which indicates that the disease free equilibrium points E_0 is globally stable.



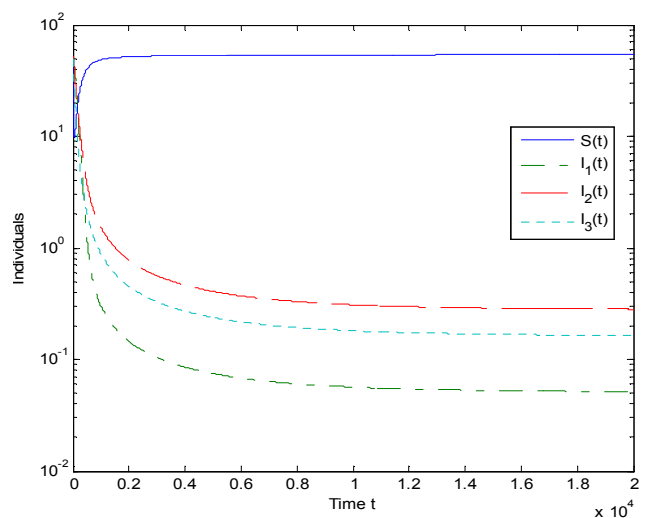
2(a)



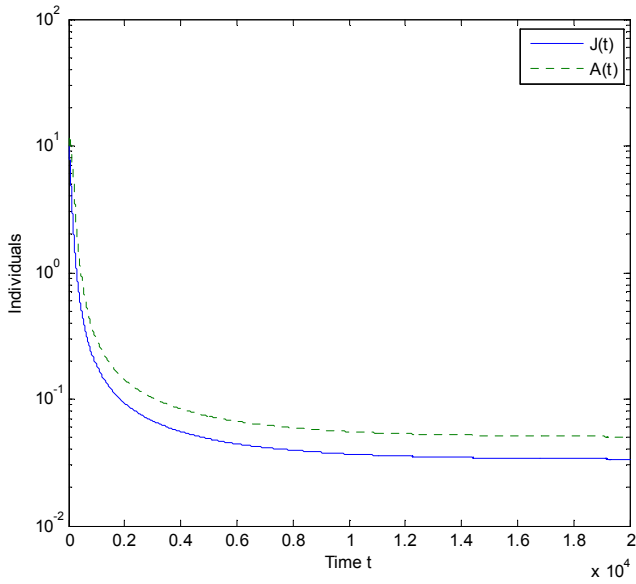
2(b)

Figs. 2. Stability of the disease free equilibrium point E_0 .

For the figs. 3(a)-3(b), the parameters are set as $p_1 = q_1 = 0.05, p_2 = q_2 = 0.9, p_3 = q_3 = 0.05, \xi_1 = 0.005, \xi_2 = 0.001$ and $\xi_3 = 0.003$. It is clear from these figs. that the reproduction number $R_0 = 1.0191 > 1$ which corresponds to the global stability of endemic equilibrium points E^* .



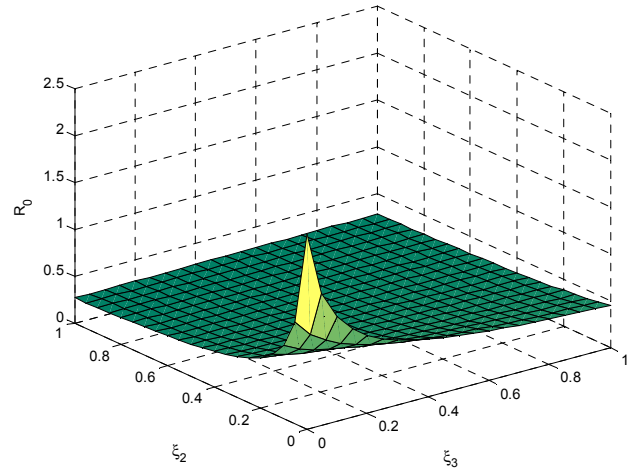
3(a)



3(b)

Figs. 3. Stability of the endemic equilibrium point E^* .

Figs. 4(a)-4(b) depict the relation among (a) R_0 & ξ_1, ξ_2 and (b) R_0 & ξ_2, ξ_3 , respectively. From these figs., it is clear that when ξ_1, ξ_2 are small and ξ_3 is too much large then the basic reproduction number R_0 of the system (7)-(12) is less than unity. Biologically, we can infer that the treatment for individuals in slow, medium and fast compartments has significant impact on the transmission dynamic of the disease.

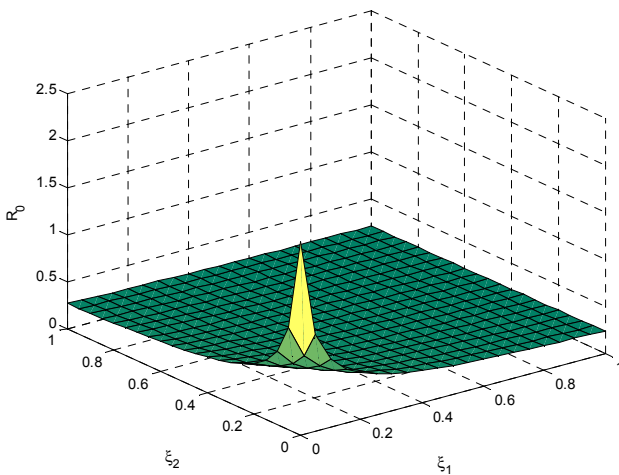


4(b)

Figs. 4. Relationship among R_0 (a) ξ_1 and ξ_2 , (b) ξ_2 and ξ_3 .

Fig. 5 depicts the relationship among R_1 , ξ_1 and ξ_2 . It is seen that when the values of ξ_1 and ξ_2 are less than R_1 of the model for slow and fast compartments, our results are in good agreement with the results of Huo and Feng [10] for two latent compartments model for which the basic reproduction number (R_1) is given by

$$R_1 = \frac{[\beta_1 b_3 (\epsilon p + b_1 (1-p)) + \beta_2 p_1 (\epsilon q + b_1 (1-q))] \frac{\Lambda}{\mu}}{b_1 b_2 b_3 - (\epsilon \xi_1 + b_1 \xi_2) p_1} \quad (45)$$



4(a)

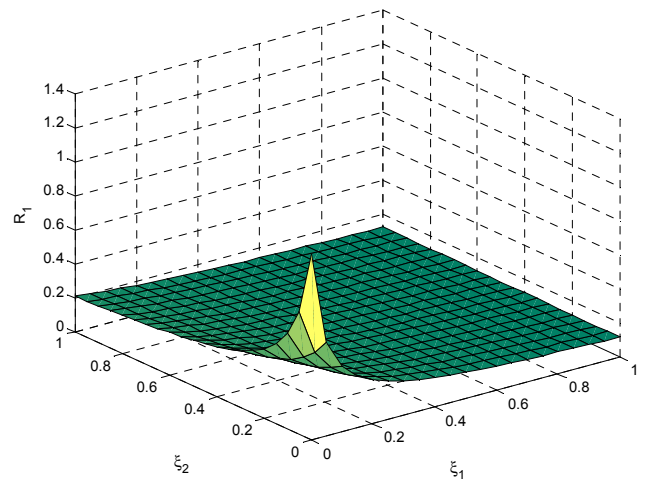
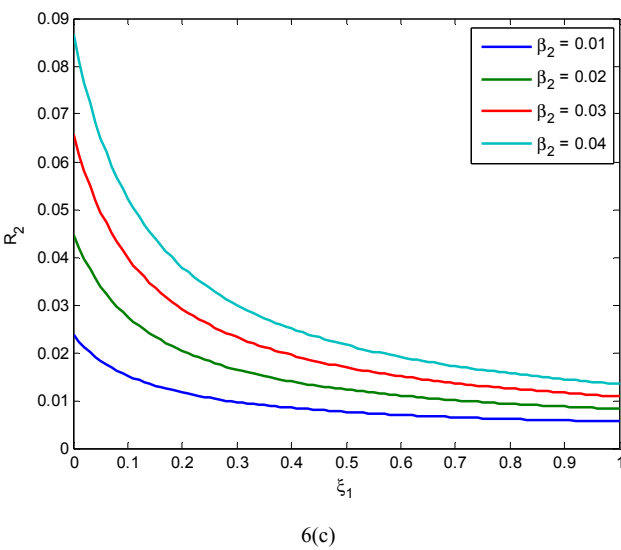
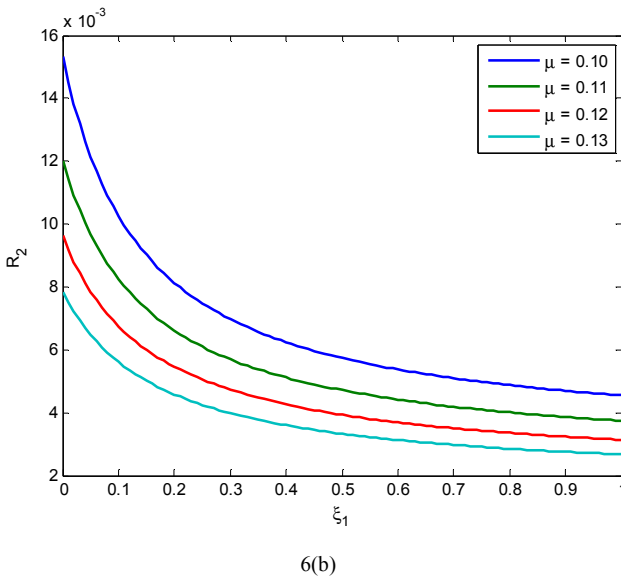
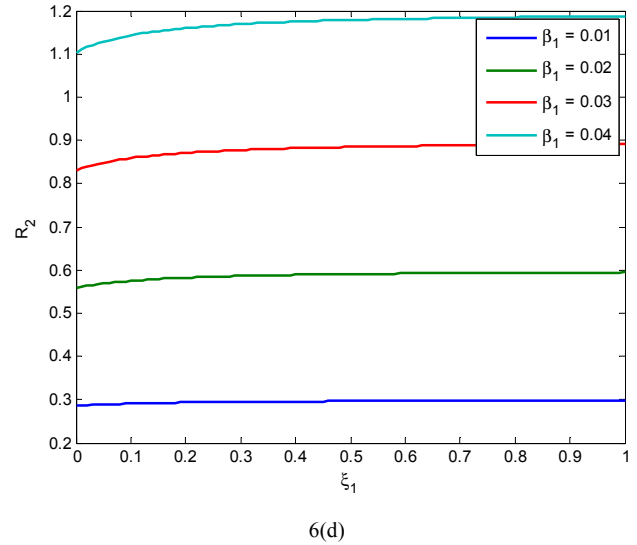
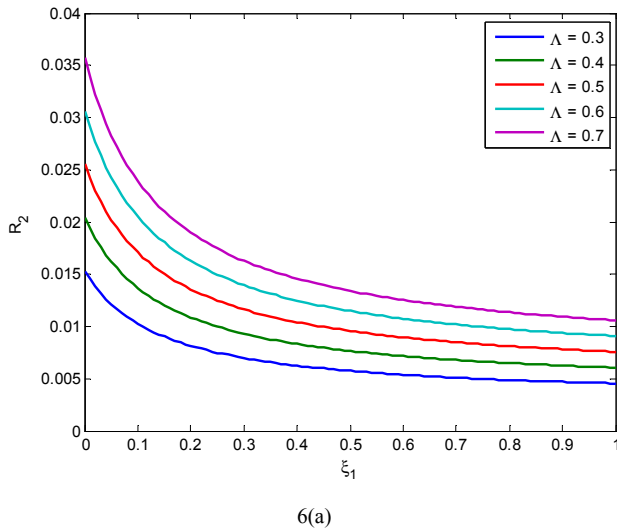


Fig. 5. Relationship among R_1 , ξ_1 and ξ_2 .



Figs. 6. Relationship among R_2 and ξ_1 for different values of (a) Λ (b) μ (c) β_1 (d) β_2 .

For varying values of parameters $\Lambda, \beta_1, \beta_2$ and μ , respectively, figs. 6(a)-6(d) exhibit the relation between

$$R_2 = \frac{[\beta_1(\mu + p_2 + \xi_1) + \beta_2 p_1] \frac{\Lambda}{\mu}}{(\mu + p_1)(\mu + p_2) + \mu \xi_1} \text{ and } \xi_1.$$

The other parameters are set as $\Lambda = 0.3, \mu = 0.1, \beta_1 = 0.0001, \beta_2 = 0.006, p_1 = 0.01$ and $p_2 = 0.03$. Fig. 6(a) displays that R_2 increases when the value of Λ goes on increasing but it decreases as ξ_1 increases. In fig. 6(b) reveals the trends of R_2 with ξ_1 for different values of μ . It is observed that the value of R_2 decreases with the increment in μ . Fig. 6(c) shows that R_2 increases by increasing β_2 . Biologically it can be interpreted easily that the transmission coefficient of symptomatic stage has significant effect on R_2 for single latent stage. Fig. 6(d) reveals that the value of R_2 remains almost constant when the value of β_1 increases; this demonstrates that the transmission coefficient of fast latent compartment has also significant effect on R_2 for single latent stage.

5. Conclusion

In this paper, the proposed epidemic model has three latent stages to explore the dynamical behavior of HIV/AIDS. Based on numerical simulation, it is concluded that the disease free equilibrium is both locally and globally asymptotically stable whenever the basic reproduction number is less than unity. Further, the endemic equilibrium is globally asymptotically stable whenever the corresponding reproduction number is greater than unity. It is noticed that the treatment of infective compartment has positive impact on HIV/AIDS control. The present model can be extended to multi-latent compartments for infective population depending

upon several stages due to treatment and socio-economic constraints.

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