On the Planarity of $G^{++}$

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Abstract: Let $G$ be a simple graph. The transformation graph $G^{++}$ of $G$ is the graph with vertex set $V(G) \cup E(G)$ in which the vertex $x$ and $y$ are joined by an edge if and only if the following condition holds: (i) $x, y \in V(G)$ and $x$ and $y$ are adjacent in $G$, (ii) $x, y \in E(G)$, and $x$ and $y$ are adjacent in $G$, (iii) one of $x$ and $y$ is in $V(G)$ and the other is in $E(G)$, and they are not incident in $G$. In this paper, it is shown $G^{++}$ is planar if and only if $|E(G)| \leq 2$ or $G$ is isomorphic to one of the following graphs: $C_3, K_1, P_4, P_4 + K_1, P_5 + K_2, P_3 + K_2 + K_1, K_{1,1}$, $3K_2, 3K_3 + K_1, 3K_2 + 2K_1, C_4, C_4 + K_1$.

Keywords: Total Graph, Planarity, Transformation Graph

1. Introduction

All graphs considered here are finite, simple and undirected. Undefined terminology and notation can be found in [2]. Let $G = (V(G), E(G))$ be a graph. $|V(G)|$ is called the order of $G$, $|E(G)|$ is called the size of $G$. The neighborhood $N_G(v)$ of $v$ is the set of all vertices of $G$ adjacent to $v$. Since $G$ is simple, $|N_G(v)| = d_G(v)$.

Suppose that $V'$ is a nonempty subset of $V(G)$. The subgraph $G[V']$ of $G$ induced by $V'$ is a graph with $V(G[V']) = V'$ and $uv \in E(G[V'])$ if and only if $uv \in E(G)$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The union $G \cup H$ of $G$ and $H$ is the graph whose vertex set is $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. Particularly, we denote their union by $G + H$ if they are disjoint, i.e. $V(G) \cap V(H) = \emptyset$.

The line graph $L(G)$ of $G$ is the graph whose vertex set is $E(G)$, and in which two vertices are adjacent if and only if they are adjacent in $G$. The total graph $G^{+++}$ of $G$ is the graph whose vertex set is $V(G) \cup E(G)$, and in which two vertices are adjacent if and only if they are adjacent or incident in $G$. Wu and Meng [9] generalized the concept of total graph, and introduced some new graphical transformations. We adopt the symbol $G^{+++}$ with $x, y, z \in \{+,−\}$ introduced in [9].

A graph is said to be embeddable in the plane, or planar, if it can be drawn in the plane so that its edges intersect only at their end vertices. A subdivision of a graph $G$ is a graph that can be obtained from $G$ by a sequence of edge subdivisions. Behzad [1] characterized the graphs $G$ for which $G^{++}$ is planar. Liu [8] give a necessary and sufficient condition for a graph $G$ for which $G^{++}$ is planar. Wu et al. [10] proved that $G^{+++}$ is planar if and only if the order of $G$ is at most 4. We refer to [4, 5, 6, 7, 10, 12, 13] for more relevant results on $G^{+++}$. As usual, the complete graph, the cycle, the path of order $n$ are denoted $K_n, C_n, P_n$, respectively.

We use the well-known theorem of Kuratowski [2] in Section 2.

Theorem 1.1. A graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$.

Corollary 1.2. Every simple planar graph has a vertex of degree at most five.

Our main result is given as follows.

Theorem 1.3. Let $G$ be a graph of size $m$. Then $G^{++}$ is planar if and only if $m \leq 2$ or $G$ is isomorphic to one of the following graphs: $C_3, C_3 + K_1, P_4, P_4 + K_1, P_5 + K_2, P_3 + K_2 + K_1, K_{1,1}$, $3K_2, 3K_3 + K_1, 3K_2 + 2K_1, C_4, C_4 + K_1$.

Proof. It is immediate form the results of Lemmas 2.1-2.5.

2. Proof

We start with a trivial observation.
Lemma 2.1. If H is a subgraph of G, then H++− is a subgraph of G++−.

In particular, by Lemma 2.1, if H++− is nonplanar and G = H + kK1 for an integer k ≥ 1, then G++− is nonplanar. One can easily check that G++− is planar for each G of size m ≤ 2.

Next we consider the graphs of size 3. There are precisely five graphs of size 3 without isolated vertex as shown in Figure 1.

**Figure 1.** All graphs of size 3 with no isolated vertices.

Lemma 2.2. For a graph G of size 3, G++− is planar if and only if G ∈ {C3, C3 + K1, P4, P4 + K1, P3 + K2, P3 + K2 + K1, K1,3 + K1, 3K2, 3K2 + K1, 3K2 + 2K1}.

**Proof.** The sufficiency. As illustration in Figure 2, the transformation graphs G++− of C3 + K1, P4 + K1, P3 + K2 + K1, K1,3 + K1, 3K2 + 2K1 are planar. By Lemma 2.1, the transformation graphs G++− of C3, P4, P3 + K2, K1,3, 3K2, 3K2 + K1 are planar.

The necessity. For each G ∈ {C3 + 2K1, P4 + 2K1, P3 + K2 + 2K1, K1,3 + 2K1, 3K2 + 3K1} the transformation graph (G + 2K1)++− of G is nonplanar since it contain a subdivision of K5 or K3,3, as shown in Figure 3.

**Figure 2.** Transformation graphs G++− of C3 + K1, P4 + K1, P3 + K2 + K1, K1,3 + K1, 3K2 + 2K1.

**Figure 3.** Transformation graphs G++− of C3 + 2K1, P4 + 2K1, P3 + K2 + 2K1, K1,3 + 2K1, 3K2 + 3K1.

Now we consider the graphs of size 4. There are precisely eleven graphs of size 4 without isolated vertex as shown in Figure 4.
Lemma 2.3. For a graph \( G \) of size 4, \( G^{++} \) is planar if and only if \( G \in \{C_4, C_4 + K_1\} \).

Proof. The sufficiency. The planar embedding of \((C_4 + K_1)^{++}\) in Figure 6 shows that \((C_4 + K_1)^{++}\) is planar. Moreover, by Lemma 2.1, \((C_4)^{++}\) is planar.

The necessity. Let \( G \) be a graph of size 4. Then \( G \) can be obtained from a graph in Fig. 4 by adding some isolated vertices. By Figure 4, 5, 6, 7 and Lemma 2.1, \( G^{++} \) is nonplanar if \( G \not\in \{C_4, C_4 + K_1\} \).

Figure 4. All graphs of size 4 with no isolated vertices.

Figure 5. Transformation graphs \( G^{++} \) of \( P_5, P_4 + K_2, P_3 + 2K_2, 4K_2 \).

Figure 6. Transformation graphs \( G^{++} \) of some graphs of size 4.
Now we consider graphs of size 5. There are precisely twenty six graphs of size 5 without isolated vertices as shown in Figure 8.

Lemma 2.4. For any graph $G$ of size 5, $G^{++}$ is nonplanar.

Proof. Let $G$ be a graph of size 5, and let $H$ be subgraph of $G$ with size 4 without isolated vertices. By Lemma 2.1, $H^{++}$ is a subgraph of $G^{++}$. By Lemma 2.3, $H^{++}$ is nonplanar if $H$ is not isomorphic to $C_4$. Now assume that $G$ contains $C_4$. Then $G$ is isomorphic to the third graph in Figure 9, and one can see that $G^{++}$ is nonplanar.
Lemma 2.5. For a graph $G$ of size $m \geq 6$, $G^{+++}$ is nonplanar.

Proof. Trivially, $G$ contains a subgraph $H$ of size 5, and by Lemma 2.1, $H^{++−}$ is a subgraph of $G^{++−}$. Furthermore, by Lemma 2.4, $G^{++−}$ is nonplanar.

3. Conclusion

In this paper, a necessary and sufficient condition for a graph $G$ such that $G^{++−}$ is planar. It is interesting to investigate some other properties or parameters, such as chromatic number, connectivity, domination number.

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References


