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Fixed Point Theorem in 2 – Metric spaces of implicit Relations

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Abstract: In this paper present on Fixed point Theorem in 2-metric spaces .A concept which has been in focus recent times. The result is supported with an example.

Keywords: Metric Spaces, Fixed Points, Complete Metric Spaces, Cauchy Sequence

1. Introduction

Theorem: -
let (X ,d) and (Y ,ρ)be a complete metric space .if T:X→Y and S :Y →X satisfying the Conditions :

1.1. [Fisher [1]]

$d(Sy , STx) \le c \max\{d(x , Sy) , d(x , STx) , \rho(y , Tx)\}$

$\rho(Tx , TSy) \le c \max\{\rho(y , Tx) , \rho(y , TSy) , d(x , Sy)\}$

for all x in X and y in Y, where $0\le c<1$,

1.2. [Popa [3]]

$d^2(Sy , STx) \le c_1 \max\{\rho(y , Tx)d(x , Sy) , \rho(y , TSy)$
$d(x , STx) , d(x , Sy)d(x , STx)\}$

$\rho^2(Tx , TSy) \le c_2 \max\{ d(x , Sy) \rho(y , Tx) , d(x , STx)$
$\rho(y , TSy) , \rho(y , Tx) \rho(y , TSy)\}$

for all x in X and y in Y, where $0\le c_1,c_2<1$.

1.3. [(Nešić[2]]

$M_1(x , y) = \{d^p(x , Sy) , d^p(x , STx) , \rho^p(y , Tx)\}$

$M_2(x , y) = \{\rho^p(y , Tx) , \rho^p(y , TSy) , d^p(x , Sy)\}$

for all x in X , y in Y and p = 1,2,3,……
let R^+ be the set of nonnegative real numbers , and let
$F_i : R^+\to R^+$ be a mapping such that
$F_i(0) = 0$ and F_i is continuous at 0 for i = 1 ,2. If T and S satisfying the inequalities

$d^p(Sy , STx) \le c_1 \max M_1(x , y) + F_1(\min M_1(x , y))$

$\rho^p(Tx , TSy) \le c_2 \max M_2(x , y) + F_2(\min M_2(x , y))$

for all x in X and y in Y, where $0\le c_1,c_2< 1$.
Then ST has a unique fixed point z in X and TS has a unique fixed point w inY .

Futher, Tz = w and Sw = z.

2. Main Results

The theorem that we are attempting to prove generalizes that Fisher [1] ,Nešić[2] , Popa [3]theorem using an implicit relations . Let $\Phi_3^{(m)}$ be the set of continuous functions with 3 variables
$\varphi: [0,\infty)^3 \to [0,\infty)$satisfying the properties:

a. φ is non decreasing in t_1 , t_2, t_3.
b. $\varphi(t,t,t) \le t^m$ $m\in N$

Some examples of such functions are as follows:

Example 2.1 $\psi(t_1, t_2, t_3) = \min\{t_1^p, t_2^p, t_3^p\}$
Example 2.2 $\psi(t_1, t_2, t_3) = \min\{t_1t_2, t_1t_3, t_2t_3\}$
Example 2.3 $\psi(t_1, t_2, t_3) = \min\{t_1, t_2, t_3\}$
Example 2.4 $\varphi(t_1, t_2, t_3) = \max\{t_1^p, t_2^p, t_3^p\}$, with m = p
Example2.5 $\varphi(t_1, t_2, t_3) = \max\{t_1, t_2, t_3\}$, with m = 1 etc.

Let F be a set of continuous functions $F :[0,\infty) \to [0,\infty)$ with F(0) = 0 (for example $F(t) = t^q$, q >0).

3. Main theorem

Theorem 3.1

Let (X,d), (Y,ρ) be two complete metric spaces and $T: X \to Y$, $S: Y \to X$ two mappings. Let $\emptyset \in \Phi_3^{(m)}$, $\psi_i \in \Psi_3$, $F_i \in f$ $for\ i = 1,2$. $if\ for\ some\ k \in [0,1)$, the following inequalities are satisfied :

$$d^m(Sy, STx) \leq k\varphi_1(d(x, STx), \rho(y,Tx)) + F_1(\psi_1(d(x, STx), \rho(y,Tx))) \quad \ldots\ldots\ldots\ldots \textbf{(1)}$$

$$\rho^m(Tx, TSy) \leq k\varphi_2(\rho(y,TSy), d(x,Sy)) + F_2(\psi_2(\rho(y,TSy), d(x,Sy))) \quad \ldots\ldots\ldots\ldots\ldots \textbf{(2)}$$

for all $x \in X$, $y \in Y$ and some $m = 1,2,\ldots\ldots$,
then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Further $T_\alpha \in \beta$ and $S_\beta \in \alpha$.

Proof :- let $x_0 \in X$ be an arbitrary point .we define the sequences (x_n) and (y_n) in X and Y respectively as follows :

$x_n = (ST)^n x_0$, $y_n = Tx_{n-1}$, $n = 1,2,3\ldots\ldots$,

Let us proof that the sequences (x_n) and (y_n) are Cauchy sequences . We assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1} \forall n \in N$, because otherwise if $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some n , we could put $\alpha = x_n$ and $\beta = y_n$.

denote

$d_n = d(x_n, x_{n+1})$, $\rho_n = \rho(y_n, y_{n+1})$, $n = 1, 2 \ldots\ldots\ldots$,

By the inequality (2) for $x = x_{n-1}$ and $y = y_n$ we get :

$$\rho^m(y_n, y_{n+1}) = \rho^m(Tx_{n-1}, TSy_n) \leq k\varphi_2(\rho(y_n, TSy_n), d(x_{n-1}, Sy_n)) + F_2(\psi_2(\rho(y_n, TSy_n), d(x_{n-1}, Sy_n)))$$

$$\rho^m(y_n, y_{n+1}) \leq k\varphi_2(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n)) + F_2(\psi_2(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n)))$$

or

$$\rho_n^m \leq k\varphi_2(\rho_n, d_{n-1}) + F_2(\psi_2(\rho_n, d_{n-1})) = k\varphi_2(\rho_n, d_{n-1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3)}$$

For the coordinates of the point (ρ_n, d_{n-1}) we have :

$\rho_n \leq d_{n-1}, \forall n \in N$, because ; in case that

$\rho_n > d_{n-1}$ for some n , if we replace the coordinates with ρ_n and apply the property (a) and (b) of φ_2 we get

$\rho_n^m \leq k\varphi_2(\rho_n, \rho_n) \leq k\,\rho_n^m$

This is impossible since $0 \leq k < 1$.

Replacing on the right hand side of (3) , the coordinates with ρ_{n-1} and applying properties (a) and (b) of φ_2 we get :

$$\rho_n^m \leq k\varphi_2(d_{n-1}, d_{n-1}) \leq k d_{n-1}^m$$

Thus $\rho_n \leq \sqrt[m]{k}\ d_{n-1}$ $\quad \ldots$ (4)

By the inequality (1) for $x = x_n$ and $y = y_n$ we get

$$d^m(x_n, x_{n+1}) = d^m(Sy_n, STx_n) \leq k\varphi_1(d(x, STx), \rho(y,Tx)) + F(\psi_1(d(x, STx), \rho(y, Tx)))$$

$$\leq k\varphi_1[d(x_n, x_{n+1}), \rho(y_n, y_{n+1})] + F_1(\psi_1(d(x_n, x_{n+1}), \rho(y_n, y_{n+1})))$$

Or

$$d_n^m \leq k\varphi_1(d_n, \rho_n) + F_1(\psi_1(d_n, \rho_n)) = k\,\varphi_1(d_n, \rho_n)$$

In similar , we get :

$d_n^m \leq k\varphi_1(\rho_n, \rho_n) \leq k\rho_n^m$

Thus we will obtain :

$d_n \leq \sqrt[m]{k}\,\rho_n$

By this inequality and (4) we get :

$$d_n \leq \sqrt[m]{k}\,(\sqrt[m]{k}\,d_{n-1}) \leq \sqrt[m]{k}\ d_{n-1} \quad \ldots\ldots\textbf{(5)}$$

by the inequalities (4) and (5) , using the mathematical induction , we obtain :

$d(x_n, x_{n+1}) = d_n \leq q^{n-1} d_i$

$\rho(y_n, y_{n+1}) = \rho_n \leq q^{n-1} d_1$ Where $q = \sqrt[m]{k} < 1$

Thus the sequences (x_n) and (y_n) are Cauchy sequences ,since the Metric space Let (X,d) ,$(Y\ \rho)$ complete metric spaces we have :

$\lim_{n\to\infty} x_n = \alpha \in X$, $\lim_{n\to\infty} y_n = \beta \in Y$

by (1) for $y = \beta$ and $x = x_n$ we get :

$$d^m(S\beta, x_{n+1}) \leq k\varphi_1(d(x_n, x_{n+1}), \rho(\beta, y_{n+1})) + F_1(\psi_1(d(x_n, x_{n+1}), \rho(\beta, y_{n+1})))$$

letting $n \to \infty$, we get :

$d^m(S\beta, \alpha) \leq k\varphi_1(0,0) + F(0) \leq k\varphi_1(0,0)$

Or

$$d^m(S\beta, \alpha) = \Leftarrow\Rightarrow S\beta = \alpha \quad \ldots\ldots (6)$$

by (2) for $x = x_n$ and $y = \beta$ we get :

$$\rho^m(y_{n+1}, TS\beta) \leq k\varphi_2(\rho(\beta, TS\beta), d(x_n, S\beta)) + F_2(\psi_2(\rho(\beta, TS\beta), d(x_n, S\beta)))$$

letting $n \to \infty$, and using(6) we get :

$\rho^m(\beta, TS\beta) \leq k\varphi_2(\rho(\beta, TS\beta), 0) + F(0)$

or

$$\rho^m(\beta, TS\beta) \leq k\rho^m(\beta, TS\beta) \Leftarrow\Rightarrow TS\beta = \beta \quad \ldots\textbf{(7)}$$

by (6) and (7) it follows :

$TS\beta = T\alpha = \beta$

$ST\alpha = S\beta = \alpha$

Thus we proved that the points α , β are fixed points of ST and TS respectively .

UNIQUENESS

suppose that ST has a second distinct Fixed point α' in X .

By (1) for $x = \alpha'$ and $y = T\alpha$ we get :

$$d^m(ST\alpha, ST\alpha') \le k\varphi_1(d(\alpha', ST\alpha'), \rho(T\alpha, T\alpha')) + F_1(\psi_1(0, \rho(T\alpha, T\alpha')))$$

Or

$$d^m(\alpha, \alpha') \le k\varphi_1 d(0, \rho(T\alpha, T\alpha')) + F_1(0) \le k\rho^m(T\alpha, T\alpha') \quad \text{..... .....} \ \textbf{(8)}$$

By (2) for $x = \alpha'$ and $y = T\alpha$ we obtain :

$$\rho^m(T\alpha', T\alpha) \le k\varphi_2(\rho(T\alpha, T\alpha), d(\alpha', \alpha)) + F_2(\psi_2(0, d(\alpha, \alpha')))$$

or

$$\rho^m(T\alpha', T\alpha) \le k\varphi_2(\rho(0, d(\alpha', \alpha)) \le kd^m(\alpha', \alpha) \quad \text{.........}\textbf{(9)}$$

By (8) and (9) we get :

$$d^m(\alpha, \alpha') \le k^2 d^m(\alpha, \alpha')$$

it follows

$$d(\alpha, \alpha') = 0$$

Thus we have again $\alpha = \alpha'$. in same way , it is proved the inequality of β ■

Hence Proved

Theorem 3.2

Let (X ,d) , (Y, ρ) be two complete metric spaces and $T : X \to Y$, $S : Y \to X$ two mappings . Let $\emptyset \in \Phi_3^{(m)}$, $\psi_i \in \Psi_3$,$F_i \in f$ $for\ i = 1, 2$. $if\ for\ some$ $k \in [0,1)$, the following inequalities are satisfied :

$$\mathbf{d^m(Sy, STx) \le k\varphi_1\Big(d(x, Sy), d(x, STx), \rho(y, Tx), \frac{d(x,Sy)\, d(x,STx)\{1+\rho(y,Tx)\}}{1+d(x,Sy)}\Big) + F_1\Big(\psi_1\Big(d(x, Sy), d(x, STx), \rho(y, Tx), \frac{d(x,Sy)\, d(x,STx)\{1+\rho(y,Tx)\}}{1+d(x,Sy)}\Big)\Big)} \quad \textbf{........ (1)}$$

$$\mathbf{\rho^m(Tx, TSy) \le k\varphi_2\Big(\rho(y, Tx), \rho(y, TSy), d(x, Sy), \frac{\rho(y,Tx)\, d(x,Sy)\{1+\rho(y,TSy)\}}{1+\rho(y,Tx)}\Big) + F_2\Big(\psi_2\Big(\rho(y, Tx), \rho(y, TSy), d(x, Sy), \frac{\rho(y,Tx)\, d(x,Sy)\{1+\rho(y,TSy)\}}{1+\rho(y,Tx)}\Big)\Big)} \quad \textbf{......... (2)}$$

for all $x \in X$, $y \in Y$ and some m = 1 ,2,……..,then ST has a unique fixed point $\alpha \in X$ and TS has a unique fixed point $\beta \in Y$. Futher $T_\alpha \in \beta$ and $S_\beta \in \alpha$.

Proof :- let $x_0 \in X$ be an arbitrary point . we define the sequences (x_n) and (y_n) in X and Y respectively as follows :

$$x_n = (ST)^n x_0 , \quad y_n = Tx_{n-1} , \quad n = 1,2,3\ldots\ldots,$$

Let us proof that the sequences (x_n) and (y_n) are Cauchy sequences .

We assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1} \forall n \in N$, because otherwise if $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some n , we could put $\alpha = x_n$ and $\beta = y_n$.

denote $d_n = d(x_n, x_{n+1})$, $\rho_n = \rho(y_n, y_{n+1})$, $n = 1, 2 \ldots\ldots\ldots,$

By the inequality (2) for $x = x_{n-1}$ and $y = y_n$ we get :

$$\rho^m(y_n, y_{n+1}) = \rho^m(Tx_{n-1}, TSy_n) \le k\varphi_2\Big(\rho(y_n, Tx_{n-1}), \rho(y_n, TSy_n), d(x_{n-1}, Sy_n), \frac{\rho(y_n, Tx_{n-1})\, d(x_{n-1}, Sy_n)\{1+\rho(y, TSy_n)\}}{1+\rho(y_n, Tx_{n-1})}\Big) + F_2\Big(\psi_2\Big(\rho(y_n, Tx_{n-1}), \rho(y_n, TSy_n), d(x_{n-1}, Sy_n), \frac{\rho(y_n, Tx_{n-1})\, d(x_{n-1}, Sy_n)\{1+\rho(y, TSy_n)\}}{1+\rho(y_n, Tx_{n-1})}\Big)\Big)$$

$$\rho^m(y_n, y_{n+1}) \le k\varphi_2\Big(\rho(y_n, y_n), \rho(y_n, y_{n+1}), d(x_{n-1}, x_n), \frac{\rho(y_n, y_n)\, d(x_{n-1}, x_n)\{1+\rho(y_n, y_{n+1})\}}{1+\rho(y_n, y_n)}\Big) + F_2\Big(\psi_2\Big(\rho(y_n, y_n), \rho(y_n, y_{n+1}), d(x_{n-1}, x_n), \frac{\rho(y_n, y_n)\, d(x_{n-1}, x_n)\{1+\rho(y_n, y_{n+1})\}}{1+\rho(y_n, y_n)}\Big)\Big)$$

or

$$\rho_n^m \le k\varphi_2(0, \rho_n, d_{n-1}, 0) + F_2(\psi_2(0, \rho_n, d_{n-1}, 0)) = k\varphi_2(0, \rho_n, d_{n-1}, 0) \quad \textbf{..........(3)}$$

For the coordinates of the point (ρ_n, d_{n-1}) we have : $\rho_n \le d_{n-1}$,$\forall n \in N$, because ; in case that

$\rho_n > d_{n-1}$ for some n , if we replace the coordinates with ρ_n and apply the property (a) and (b) of φ_2 we get

$$\rho_n^m \le k\varphi_2(\rho_n, \rho_n, \rho_n, \rho_n) \le k\, \rho_n^m$$

This is impossible since $0 \le k < 1$.

Replacing on the right hand side of (3) , the coor1dinates with ρ_{n-1} and applying properties (a) and (b) of φ_2 we get :

$$\rho_n^m \le k\varphi_2(d_{n-1}, d_{n-1}, d_{n-1}, d_{n-1}) \le k d_{n-1}^m$$

Thus

$$\rho_n \le \sqrt[m]{k}\ d_{n-1} \quad \textbf{............... (4)}$$

By the inequality (1) for $x = x_n$ and $y = y_n$ we get

$$d^m(x_n, x_{n+1}) = d^m(Sy_n, STx_n)$$

$\leq k\varphi_1(d(x_n,Sy_n), d(x_n,STx_n), \rho(y_n,Tx_n), \frac{d(x_n,Sy_n)\ d(x_n,STx_n)\{1+\rho(y_n,Tx_n)\}}{1+d(x_n,Sy_n)})$

$+ F_1(\psi_1(d(x_n,Sy_n), d(x_n,STx_n), \rho(y_n,Tx_n), \frac{d(x_n,Sy_n)\ d(x_n,STx_n)\{1+\rho(y_n,Tx_n)\}}{1+d(x_n,Sy_n)}))$

$\leq k\varphi_1(d(x_n,x_n), d(x_n,x_{n+1}), \rho(y_n,y_{n+1}), \frac{d(x_n,x_n)\ d(x_n,x_{n+1})\{1+\rho(y_n,y_{n+1})\}}{1+d(x_n,x_n)})$

$+ F_1(\psi_1(d(x_n,x_n), d(x_n,x_{n+1}), \rho(y_n,y_{n+1}), \frac{d(x_n,x_n)\ d(x_n,x_{n+1})\{1+\rho(y_n,y_{n+1})\}}{1+d(x_n,x_n)}))$

Or

$d_n^m \leq k\varphi_1(0, d_n, \rho_n, 0) + F_1(\psi_1(0, d_n, \rho_n, 0))$

$= k\varphi_1(0, d_n, \rho_n, 0)$

In similar , we get :

$d_n^m \leq k\varphi_1(\rho_n, \rho_n, \rho_n, \rho_n) \leq k\rho_n^m$

Thus we will obtain :

$d_n \leq \sqrt[m]{k}\,\rho_n$

By this inequality and (4) we get :

$d_n \leq \sqrt[m]{k}\,(\sqrt[m]{k}\,d_{n-1}) \leq \sqrt[m]{k}\ d_{n-1}$ **....... (5)**

by the inequalities (4) and (5) , using the mathematical induction , we obtain :

$d(x_n, x_{n+1}) = d_n \leq q^{n-1} d_i$

$\rho(y_n, y_{n+1}) = \rho_n \leq q^{n-1} d_1$

Where $q = \sqrt[m]{k} < 1$

Thus the sequences (x_n) and (y_n) are Cauchy sequences ,since the Metric space Let (X,d) ,(Y,ρ) complete metric spaces we have :

$\lim_{n\to\infty} x_n = \alpha \in X$, $\lim_{n\to\infty} y_n = \beta \in Y$

by (1) for $y=\beta$ and $x = x_n$ we get :

$d^m(S\beta, x_{n+1}) \leq k\varphi_1(d(x_n,S\beta), d(x_n,x_{n+1}), \rho(\beta,y_{n+1}), \frac{d(x_n,S\beta)d(x_n,x_{n+1})\{1+\rho(\beta,y_{n+1})\}}{1+d(x_n,S\beta)})$

$+ F_1(\psi_1(d(x_n,S\beta), d(x_n,x_{n+1}), \rho(\beta,y_{n+1}), \frac{d(x_n,S\beta)d(x_n,x_{n+1})\{1+\rho(\beta,y_{n+1})\}}{1+d(x_n,S\beta)}))$

letting $n\to\infty$, we get :

$d^m(S\beta,\alpha) \leq k\varphi_1(d(\alpha,S\beta), 0,0,0) + F(0)$

$\leq k\varphi_1(d(\alpha,S\beta)\ 0,0,0) \leq kd^m(\alpha,S\beta)$

Or

$d^m(S\beta,\alpha) = 0 \iff S\beta = \alpha$ **......... (6)**

by (2) for $x = x_n$ and $y = \beta$ we get :

$\rho^m(y_{n+1}, TS\beta) \leq k\varphi_2(\rho(\beta,Tx_n), \rho(\beta,TS\beta), d(x_n,S\beta), \frac{\rho(\beta,Tx_n)d(x_n,S\beta)\{1+\rho(\beta,TS\beta)\}}{1+\rho(\beta,Tx_n)})$

$+ F_2(\psi_2(\rho(\beta,Tx_n), \rho(\beta,TS\beta), d(x_n,S\beta), \frac{\rho(\beta,Tx_n)d(x_n,S\beta)\{1+\rho(\beta,TS\beta)\}}{1+\rho(\beta,Tx_n)}))$

$\rho^m(y_{n+1}, TS\beta) \leq k\varphi_2(\rho(\beta,y_{n+1}), \rho(\beta,TS\beta), d(x_n,S\beta), \frac{\rho(\beta,y_{n+1})d(x_n,S\beta)\{1+\rho(\beta,TS\beta)\}}{1+\rho(\beta,y_{n+1})})$

$+ F_2[\psi_2(\rho(\beta,y_{n+1}), \rho(\beta,TS\beta), d(x_n,S\beta), \frac{\rho(\beta,y_{n+1})d(x_n,S\beta)\{1+\rho(\beta,TS\beta)\}}{1+\rho(\beta,y_{n+1})}))$

letting $n\to\infty$, and using(6) we get :

$\rho^m(\beta,TS\beta) \leq k\varphi_2(0, \rho(\beta,TS\beta), 0, 0) + F(0)$

or

$\rho^m(\beta,TS\beta) \leq k\rho^m(\beta,TS\beta) \iff TS\beta = \beta$

.... (7)

by (6) and (7) it follows :

$TS\beta = T\alpha = \beta$

$ST\alpha = S\beta = \alpha$

Thus we proved that the points α , β are fixed points of ST and TS respectively .

UNIQUENESS

suppose that ST has a second distinct fixed point α' in X . By (1) for $x = \alpha'$ and $y = T\alpha$ we get :

$d^m(ST\alpha, ST\alpha')$

$\leq k\varphi_1(d(\alpha',ST\alpha), d(\alpha',ST\alpha'), \rho(T\alpha,T\alpha'), \frac{d(\alpha',ST\alpha)\ d(\alpha',ST\alpha')\{1+\rho(T\alpha,T\alpha')\}}{1+d(\alpha',ST\alpha)})$

$+ F_1(\psi_1(d(\alpha',ST\alpha), d(\alpha',ST\alpha'), \rho(T\alpha,T\alpha'), \frac{d(\alpha',ST\alpha)\ d(\alpha',ST\alpha')\{1+\rho(T\alpha,T\alpha')\}}{1+d(\alpha',ST\alpha)}))$

$d^m(ST\alpha, ST\alpha') \leq k\varphi_1(d(\alpha',\alpha), 0, \rho(T\alpha,T\alpha'), 0)$
$+ F_1(\psi_1(d(\alpha',\alpha), 0, \rho(T\alpha,T\alpha'), 0))$

Or

$d^m(\alpha,\alpha') \leq k\varphi_1(d(\alpha',\alpha), 0, \rho(T\alpha,T\alpha'), 0)$
$+ F_1(0)$
$\leq k\rho^m(T\alpha,T\alpha')$ **..........(8)**

By (2) for $x = \alpha'$ and $y = T\alpha$ we obtain :

$\rho^m(T\alpha',T\alpha) \leq k\varphi_2(\rho(T\alpha,T\alpha'), \rho(T\alpha,T\alpha), d(\alpha',\alpha), \frac{\rho(T\alpha,T\alpha')\,d(\alpha',\alpha)\{1+\rho(T\alpha,T\alpha)\}}{1+\rho(T\alpha,T\alpha')})$

$+ F_2(\psi_2(\rho(T\alpha\ T\alpha'), \rho(T\alpha,T\alpha), d(\alpha,\alpha'), \frac{\rho(T\alpha,T\alpha')\,d(\alpha',\alpha)\{1+\rho(T\alpha,T\alpha)\}}{1+\rho(T\alpha,T\alpha')}))$

$\rho^m(T\alpha',T\alpha) \leq k\varphi_2(\rho(T\alpha,T\alpha'), 0, d(\alpha',\alpha), \frac{\rho(T\alpha,T\alpha')\,d(\alpha',\alpha)\{1+0\}}{1+\rho(T\alpha,T\alpha')})$

$+ F_2(\psi_2(\rho(T\alpha,T\alpha'), 0, d(\alpha,\alpha'), \frac{\rho(T\alpha,T\alpha')\,d(\alpha',\alpha)\{1+0\}}{1+\rho(T\alpha,T\alpha')}))$

or

$$\rho^m(T\alpha', T\alpha) \leq k\varphi_2(\rho(T\alpha', T\alpha), 0, d(\alpha, \alpha'), \frac{\rho(T\alpha, T\alpha')\, d(\alpha', \alpha)}{1+\rho(T\alpha, T\alpha')})$$
$$+ F_2(0)$$
$$\leq kd^m(\alpha, \alpha')\rho^m(T\alpha', T\alpha)$$
$$\leq k\varphi_2(\rho(T\alpha', T\alpha), 0, d(\alpha, \alpha'), \frac{\rho(T\alpha, T\alpha')\, d(\alpha', \alpha)}{1+\rho(T\alpha, T\alpha')})$$
$$\leq kd^m(\alpha, \alpha') \qquad \textbf{........(9)}$$

By (8) and (9) we get :

$$d^m(\alpha, \alpha') \leq k^2 \, d^m(\alpha, \alpha')$$

it follows

$d(\alpha, \alpha') = 0$Thus we have again $\alpha = \alpha'$. in same way , it is proved the inequality of β ■

Hence Proved

References

[1] B.Fisher,Fixedpointontwometricspaces,GlasnikMathem.16(36)(1981),333-337.

[2] S.Č.Nešić,Afixedpointtheoremintwometricspaces,Bull.Math.Soc.Sci.Math.Roumanie,Tome44(94)(2001)No.3,253-257.

[3] V.Popa,Fixedpointsontwocompletemetricspaces,Zb.Rad.Prirod.-Mat.Fak.Sez.Mat.21,1(1991),83-93.