



# A fixed point theorem on reciprocally continuous self mapping under menger space

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**Abstract:** The Fixed Point Theorem had been proved on Reciprocally Continuous Self Mapping. In this paper the fixed point theorem on reciprocally continuous self mapping is proved under Menger Space.

**Keywords:** Fixed Point, Reciprocally Continuous, Compatible Maps, Self Maps, Complete Menger Space

## 1. Introduction

In 1964 Metric spaces were introduced by Gähler, the probabilistic metric spaces is an important part of stochastic Analysis, to develop the fixed point theory in such spaces. There are many results in fixed point theory in probabilistic metric space., since then there have been many fixed point theorems proved in metric spaces and as a generalization of metric spaces, there have been only a few results in fixed point theory.

A coincidence point theorem for multi valued mappings satisfying generalized Hicks contraction principle in Menger Space. A probabilistic metric space is introduced by Menger. Many fixed point results have been obtained for single valued in probabilistic metric Spaces. Fixed Point theorem is proved for multi-valued version and by using the and by using the notion of the function of non compactness, notion of the function of non compactness, A multi-valued generalization of the notion of a contraction and Fixed Point theorem are introduced in Hadzic generalized Fixed Point theorem for multi-valued in zikic proved a coincidence point theorem for three mappings which is a generalization of Hicks theorem.

Fixed point theory in Menger spaces is a developed branch of mathematics. Sehgal and Bharucha-Reid First introduced the contraction mapping principle in probabilistic metric spaces [Hadzic and Pap]

## 2. Preliminaries

### 2.1. Definition: [18]

A t-norm is a function  $\Delta: [0,1] \times [0,1] \rightarrow [0,1]$  which satisfies the following conditions.

- $\Delta(1, a) = a$
- $\Delta(a, b) = \Delta(b, a)$
- $\Delta(c, d) \geq \Delta(a, b)$  whenever  $c \geq a$  and  $d \geq b$
- $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$

### 2.2. Definition

A mapping  $F: R \rightarrow R^+$  is called a distribution if it is non-decreasing and left continuous with  $\inf_{t \in R} F(t) = 0$ ,  $\sup_{t \in R} F(t) =$

1, where  $R$  is the set of real numbers and  $R^+$  denotes the set of non-negative real numbers.

### 2.3. Definition Menger Space [18]

A Menger Space is a triplet  $(M, F, \Delta)$  where  $M$  is a non-empty set,  $F$  is a function defined on  $M \times M$  to the set of distribution functions and  $\Delta$  is a t-norm, such that the following are satisfied.

- $F_{x,y}(0) = 0$  for all  $x, y \in M$ ,
- $F_{x,y}(s) = 1$  for all  $s > 0$  and  $x, y \in M$ , if

and only if  $x=y$

iii.  $F_{x,y}(s) = F_{y,x}(s)$  for all  $x, y \in M$ ,  $s > 0$

and

iv.  $F_{x,y}(u+v) \geq \Delta(F_{x,z}(u), F_{x,y}(v))$  for all  $u, v \geq 0$  and

$x, y, z \in M$

A sequence  $\{X_n\} \subset M$  converges to some point  $x \in M$  if for given  $\varepsilon \in M$  if for given  $\varepsilon > 0$ ,  $\lambda > 0$  we can find a positive integer  $N_{\varepsilon, \lambda}$  such that for all  $n > N_{\varepsilon, \lambda}$ .

$$F_{x_n, x}(x) > 1 - \lambda$$

## 2.4. Definition: (Cauchy Sequence)

A sequence  $\{x_n\}$  in a Menger space  $(M, F, \Delta)$  is called a Cauchy sequence if for each  $\varepsilon \in (0, 1)$  and  $t > 0$ . There exists  $\eta_0 \in N$  such that  $F_{x_n, x_m}(t) > 1 - \varepsilon$  for all  $m, n \geq \eta_0$ . The Menger space  $(M, F, \Delta)$  is said to be complete if every Cauchy sequence in  $M$  convergent.

### Lemma 2.1

Let  $X$  be metric space and  $(X, F, \Delta)$  be a Menger probabilistic metric space with metric  $d$  and let  $w$  be  $w$ -distance,  $t$ -norm in  $\Delta$  on  $x$ . Let  $\{x_n\}$  and  $\{y_n\}$  be a sequence in  $X$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be a sequence in  $(0, \infty)$  converging to 0, and let  $x, y, z \in X$ . Then, the following hold:

- If  $w(x_n, y) \leq \alpha_n$  and  $w(x_n, Z) \leq \beta_n$  for any  $n \in N$ , then  $y=z$ , in particular, if  $w(x, y) = 0$  and  $w(x, z) = 0$  then  $y=z$ ,
- If  $w(x_n, y_n) \leq \alpha_n$  and  $w(x_n, Z) \leq \beta_n$  for  $n \in N$ , then  $\{y_n\}$  converges to  $z$ ;
- If  $w(x_n, x_m) \leq \alpha_n$  for any  $n, m \in N$  with  $m > n$ , then  $\{x_n\}$  is a Cauchy sequence;
- If  $w(y, x_n) \leq \alpha_n$  for any  $n \in N$ , then  $\{x_n\}$  is a Cauchy sequence.

## 3. Main Result

### Theorem

Let  $S, I, T, J$  be self maps of a complete Menger space  $(X, F, \Delta)$  with  $t\Delta t \geq t \forall t \in [0, 1]$

- $S(x) \subset J(x), T(x) \subset I(x)$

$$d_{S_x, T_y}^2 \leq \max \left\{ \phi(d_{I_x, J_y}), \phi(d_{I_x, S_x}), \phi(d_{I_x, J_y}), \phi(d_{J_y, T_y}), \right.$$

$$\left. \phi(d_{I_x, S_x}), \phi(d_{J_y, T_y}), \phi(d_{I_x, J_y}), \phi(d_{J_y, S_x}) \right\}$$

for  $\forall x, y \in X$

- $SI=IS, ST=TS$

- $SI$  or  $S$  is continuous

- The pair  $(S, I)$   $(T, J)$  weak compatible and compatible.

Then  $S, I, T$  &  $J$  have unique common fixed point.

### Proof

Let  $x_0$  in  $X$  be arbitrary, choose a point  $x_1$  in  $X$  such that  $Sx_0 = Jx_1$ . This can be done since  $S(x) \subset J(x)$ . Let  $x_2$  be a point in  $X$  such that  $Tx_1 = Ix_2$ . This can be done since  $T(x) \subset I(x)$ .

In general we can choose  $x_{2n}, x_{2n+1}, x_{2n+2}, \dots$  such that  $Sx_{2n} = Jx_{2n+1}$  and  $Tx_{2n+1} = Ix_{2n+2}$ , so that we obtain a sequence

$$Sx_0, Tx_1, Tx_3 \quad (1)$$

Taking condition (i), (ii) and (iii) as in Aage and Salunke [1]

$\{Sx_{2n}\}$  is a Cauchy sequence and consequently the sequence (1) is a Cauchy. The sequence (1) converges to a limit  $z$  in  $X$ . Hence the subsequence  $\{Sx_{2n}\} = \{Jx_{2n+1}\}$  and  $\{Tx_{2n+1}\} = \{Ix_{2n+2}\}$  also converges to the limit point  $z$ .

Suppose that the pair  $(S, I)$  is compatible pair of reciprocally continuous. By the definition of reciprocally continuous, there is a sequence  $\langle x_n \rangle$  in  $X$  such that

$$Sx_{2n} \rightarrow z, Ix_{2n} \rightarrow z, \text{ then } SIx_{2n} \rightarrow Sz$$

$$ISx_{2n} \rightarrow Iz, n \rightarrow \infty \quad (2)$$

Since the pair  $(S, I)$  is compatible we have  $Sx_{2n} \rightarrow z, Ix_{2n} \rightarrow z$ , as  $n \rightarrow \infty$  and

$$\lim_{n \rightarrow \infty} (d_{SIx_{2n}, ISx_{2n}}) = 0 \quad (3)$$

Using (2) and (3) we get

$$d(Sz, Iz) = 0 \text{ or } Sz = Iz$$

$$\text{since } Sz = Iz$$

Now

$$d_{Sx_{2n+1}, Tx_{2n+1}}^2 \leq \max \left\{ \phi(d_{Iz, Jx_{2n+1}}), \phi(d_{Iz, Sz}), \phi(d_{Iz, Jx_{2n+1}}), \phi(d_{Jx_{2n+1}, Tx_{2n+1}}), \right. \\ \left. \phi(d_{Iz, Sz}), \phi(d_{Jx_{2n+1}, Tx_{2n+1}}), \phi(d_{Iz, Tx_{2n+1}}), \phi(d_{Jx_{2n+1}, Sz}) \right\}$$

$$Jx_{2n+1} \rightarrow z, Tx_{2n+1} \rightarrow z \text{ as } n \rightarrow \infty \text{ and } Iz = Sz, \text{ so letting } n \rightarrow \infty \text{ we get}$$

$$d_{Sz, z}^2 \leq \max \left\{ \phi(d_{Sz, z}), \phi(d_{Sz, z}), \phi(d_{Sz, z}), \phi(d_{z, z}), \right. \\ \left. \phi(d_{Sz, Sz}), \phi(d_{z, z}), \phi(d_{Sz, z}), \phi(d_{z, Sz}) \right\}$$

$$\text{i.e. } d_{Sz, z} \leq \phi(d_{Sz, z}) \leq d_{Sz, z} \text{ Hence}$$

$$\phi(d_{Sz, z}) = 0 \text{ i.e. } Sz = z \text{ Thus } Sz = Tz = z$$

Further since  $S(x) \subset J(x)$ , There is a point  $w \in X$

so to  $z = Sz = Jw$

Now we prove that  $Jw = Tw$ , Now

$$d_{S_z, w}^2 \leq \max \left\{ \begin{array}{l} \phi(d_{I_z, J_z})\phi(d_{I_z, S_z}), \\ \phi(d_{I_z, J_z})\phi(d_{J_w, T_w}), \\ \phi(d_{I_z, S_z})\phi(d_{J_w, T_w}), \\ \phi(d_{I_z, T_w}), \phi(d_{J_w, S_z}) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \phi(d_{J_w, J_w})\phi(d_{J_w, J_w}), \phi(d_{J_w, J_w})\phi(d_{J_w, T_w}), \\ \phi(d_{J_w, J_w})\phi(d_{J_w, T_w}), \phi(d_{J_w, T_w}), \phi(d_{J_w, J_w}) \end{array} \right\}$$

so  $d_{J_w, T_w}^2 \leq 0$  implies  $d_{J_w, T_w} = 0$  hence

$$z = Jw = Tw$$

Since the pair (T, J) is compatible pair of reciprocally continuous. By the definition of reciprocally continuous, there is a sequence  $\langle x_n \rangle$  in X such that

$$Tx_{2n} \rightarrow z, Jx_{2n} \rightarrow z$$

$$\text{Then } TJx_{2n} \rightarrow Tz, JT x_{2n} \rightarrow Jz \text{ as } n \rightarrow \infty \quad (4)$$

Since the pair (T, J) is compatible we have  $Tx_{2n} \rightarrow z, Jx_{2n} \rightarrow z$  as  $n \rightarrow \infty$  and

$$\lim_{n \rightarrow \infty} d_{Tx_{2n}, JT x_{2n}} = 0 \quad (5)$$

Using (4) and (5) we get

$$T_{I_z, J_z} = 0 \text{ hence } Tz = Jz \text{ Now}$$

$$d_{z, Tz}^2 = d_{S_z, Tz}^2$$

$$\leq \max \left\{ \begin{array}{l} \phi(d_{I_z, J_z})\phi(d_{I_z, S_z}), \phi(d_{I_z, J_z})\phi(d_{J_z, Tz}), \\ \phi(d_{I_z, S_z})\phi(d_{J_z, Tz}), \phi(d_{I_z, Tz}), \phi(d_{J_z, S_z}) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \phi(d_{z, J_z})\phi(d_{z, z}), \phi(d_{z, Tz})\phi(d_{Tz, Tz}), \\ \phi(d_{z, z})\phi(d_{Tz, Tz}), \phi(d_{z, Tz})\phi(d_{Tz, Tz}) \end{array} \right\}$$

$$= \phi(d_{Tz, z})\phi(d_{Tz, z})$$

implies that

$$d_{Tz, z} \leq \phi(d_{Tz, z}) \leq (d_{Tz, z})$$

Hence  $\phi(d_{Tz, z}) = 0$  i.e.  $Tz = z$  and  $z = Tz = Jz$ . so  $z$  is a common fixed point of S, I, J & T.

### Uniqueness

Let  $z'$  be another common fixed point of S, I, J and T i.e.  $z' = Sz' = Iz' = Tz' = Jz'$ , from condition (ii) we have

$$d_{z, z'}^2 = d_{S_z, Tz'}^2$$

$$\leq \max \left\{ \begin{array}{l} \phi(d_{I_z, J_z'})\phi(d_{I_z, S_z}), \phi(d_{I_z, J_z'})\phi(d_{J_z', Tz'}) \\ \phi(d_{I_z, S_z})\phi(d_{J_z', Tz'}), \phi(d_{I_z, Tz'}), \phi(d_{J_z', S_z}) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \phi(d_{z, z'})\phi(d_{z, z}), \phi(d_{z, z'})\phi(d_{z', z'}), \\ \phi(d_{z, z'})\phi(d_{z', z'}), \phi(d_{z, z'}), \phi(d_{z', z}) \end{array} \right\}$$

Therefore  $d_{z, z'} \leq \phi(d_{z, z'}) \leq d_{z, z'}$  i.e.  $\phi(d_{z, z'}) = d_{z, z'}$  thus

$d_{z, z'} = 0$  i.e.  $z' = z$ . Hence the common fixed point is unique.

Hence Proved

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