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New Improved Approximation by Linear Combination in L_p Spaces

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Abstract: In this paper we extend our studies for Modified Lupas operators introduced by Sahai and Prasad. We introduce and develop some direct results for Stancu type generalization of above operators using linear approximation method. Fubini's theorem is used extensively to prove our main theorem. The anticipated improvement is made through technique of linear combination is well corroborated by the results in the paper. Here, modification of operators through Stancu generalization plays an important role to obtain better approximation results.

Keywords: Stancu Type Generalization, Linear Combination, Order of Approximation

1. Introduction

Modified Lupas Operators introduced by Sahai and Prasad [1] is

$$C_n(f,u) = \int_0^{\infty} M(n,u,t)f(t)dt \tag{1.1}$$

Where $f \in L_P[0,\infty)$ and $u \geq 1, u \in [0,\infty)$.

Stancu type generalization [2] [3] of above operator gives,

$$C_{(n)}^{(\alpha,\beta)}(f,u) = \int_0^{\infty} M(n,u,t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt \tag{1.2}$$

Where $u \in [0,\infty)$

Here $M(n,u,t) = (n-1)\sum_{m=0}^{\infty} g_{nm}(u)g_{nm}(t)$

And $g_{nm}(u) = \binom{n+m-1}{m}u^m(1+u)^{-(n+m)}$

Howsoever, smooth [4] the function may be, the order of approximation by these operators is at its best at order $O(n^{-1})$. To improve order of approximation, May [2] and Rathore [5] proposed technique of linear combination of these linear positive operators.

Let $c_0, c_1, c_2 \ldots\ldots\ldots\ldots c_n$ be $k+1$ arbitrary but fixed distinct positive integers. Then linear combination $C_{(n)}^{(\alpha,\beta)}(f,k,u)$ of $C_{(c_jn)}^{(\alpha,\beta)}(f,u)$, $j = 0(1), k$ is defined by,

$$C_{(n)}^{(\alpha,\beta)}(f,k,u) = \frac{1}{\Delta}\begin{vmatrix} C_{(c_0n)}^{(\alpha,\beta)}(f,u) & c_0^{-1} & c_0^{-2} & \ldots\ldots\ldots c_0^{-k} \\ C_{(c_1n)}^{(\alpha,\beta)}(f,u) & c_1^{-1} & c_1^{-2} & \ldots\ldots\ldots c_1^{-k} \\ \ldots\ldots\ldots\ldots\ldots & \ldots. & \ldots. & \ldots\ldots\ldots\ldots \\ C_{(c_kn)}^{(\alpha,\beta)}(f,u) & c_k^{-1} & c_k^{-2} & \ldots\ldots\ldots c_k^{-k} \end{vmatrix} \tag{1.3}$$

Where Δ is Vandermonde determinant obtained by replacing the operator column of above determinant by entries 1.

Simplification of (1.3) leads to,

$$C_{(n)}^{(\alpha,\beta)}(f,k,u) = \sum_{j=0}^{k} D(j,k)C_{(c_jn)}^{(\alpha,\beta)}(f,u) \tag{1.4}$$

Where $D(j.k) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{c_j}{c_j - c_i}$ $\quad k \neq 0$ and $D(0,0) = 1$

Let $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$ and $I_i = [a_i, b_i]$, $i = 1,2,3$. Also let $[\alpha]$ denote integral part of α.

Let $1 \leq p \leq \infty, f \in L_p[0,\infty)$. Then for sufficiently small $\eta > 0$, the steklov mean $f_{\eta,m}$ of mth order corresponding to f is defined as,

$$f_{\eta,m}(t) = \eta^{-m}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\ldots\ldots\ldots\int_{-\pi/2}^{\pi/2}\left[f(t) + (-1)^{m-1}\Delta^m \sum_{i=1}^{mt_i} f(t)\right]dt_1dt_2\ldots\ldots dt_m \tag{1.5}$$

We will use the following results:

a) $f_{\eta,m}$ has derivatives upto order m, $f_{\eta m}^{(m-1)} \in AC[I_1]$ and $f_{\eta,m}^{(m)}$ exists a.e and belongs to $L_p(I_1)$.

b) $\left\|f_{\eta,m}^{(r)}\right\|_{L_p(I_2)} \leq G_r\eta^{-r}\omega_r, r = 1,2,3,\ldots\ldots, m.$

c) $\left\|f - f_{\eta,m}\right\|_{L_p(I_2)} \leq G_{m+1}\omega_m(f,\eta,p,I_1)$

d) $\left\|f_{\eta,m}\right\|_{L_p(I_2)} \leq G_{m+2}\|f\|_{L_p(I_1)}$

e) $\|f_{\eta,m}^{(m)}\|_{L_p(I_2)} \leq G_{m+3}\eta^{-m}\|f\|_{L_p(I_1)}$

Where $G_i's$ are certain constants independent of f and η.

Here, $AC[a,b]$ represents absolute continuous function on $[a,b]$. Set of all functions of bounded variation on $[a,b]$ is represented by $BV[a,b]$. The semi-norm $\|f\|_{BV[a,b]}$ is defined by total variation of f on $[a,b]$.For $f \in L_p[a,b]$, $1 \leq p \leq \infty$, the Hardy-Little Wood majorant of f is defined as,

$$h_f(x) = \sup_{\zeta \to x} \frac{1}{\zeta - x}\int_x^\zeta f(t)dt, (a \leq \zeta \leq b)$$

2. Some Auxiliary Results

Lemma1. [1] For the function, the mth order moment $T_{(n)}^{(\alpha,\beta)}$ be defined on [0,∞) as,

$$T_{(n)}^{(\alpha,\beta)} = c_{(n)}^{(\alpha,\beta)}\left(\left(\frac{nt+\alpha}{n+\beta} - u\right)^m, u\right) \tag{2.1}$$

i.e., $T_{(n)}^{(\alpha,\beta)} = \int_0^\infty M(n,u,t)\left(\frac{nt+\alpha}{n+\beta} - u\right)^m dt$

or, $T_{(n)}^{(\alpha,\beta)} = (n-1)\sum_{m=0}^\infty g_{nm}(u)\int_0^\infty g_{nm}(t)\left(\frac{nt+\alpha}{n+\beta} - u\right)^m dt$

Here,$T_{(n)}^{(\alpha,\beta)}$ is a polynomial of degree m.

Also, $T_{(n)}^{(\alpha,\beta)}$is a rational function in n.

Consequently for $u \geq 0, T_{(n,m)}^{(\alpha,\beta)}(u) = O\left(n^{-[(m+1)/2]}\right)$, for $u \in [0,\infty)$. (2.2)

Using, Holder's inequality, we get,

$$C_{(n)}^{(\alpha,\beta)}(|t-u|^r, t) = O\left(n^{-r/2}\right) \text{ for each } r > 0 \text{ and for fixed } t \in [0,\infty). \tag{2.3}$$

Here $[\alpha]$ is an integral part of α. Also, for any given number $\lambda > 2$, and $u > 0$, there is an integer $N(\lambda,u) > 2$, such that,

$$T_{(n,2)}^{(\alpha,\beta)} \leq \frac{\lambda u(1+u)}{n} \quad \text{for all } n \in N(\lambda,u) \tag{2.4}$$

Lemma 2. For $p \in N$ and n sufficiently large there hold,

$$C_{(n)}^{(\alpha,\beta)}[(t-u)^p, k, u] = \frac{1}{n^{(k+1)}}\{S(p,k,u) + O(1)\}$$

Where $S(p,k,u)$ is certain polynomial in u of degree $p/2$.

Proof. From Lemma 1, for sufficiently large n, we can write,

$$C_{(n)}^{(\alpha,\beta)}[(t-u)^p, u] = Q_0(u)n^{-[(p+1)/2]} + Q_1(u)n^{-\{[(p+1)/2]+1\}} + \dots + Q_{[p/2]}n^{-p} + \dots$$

Where $Q_i(u), i = 0,1, \dots\dots\dots$ are certain polynomials in u of degree atmost p.

Therefore, $C_{(n)}^{(\alpha,\beta)}((t-u)^p, k, u)$ is given by,

$$\frac{1}{\Delta}\begin{vmatrix} Q_0(u)(d_0 n)^{-[(p+1)/2]} + Q_1(u)(d_0 n)^{-\{[(p+1)/2]+1\}} + \dots\dots\dots + d_0^{-1}d_0^{-2}\dots\dots d_0^{-k} \\ Q_0(u)(d_1 n)^{-[(p+1)/2]} + Q_1(u)(d_1 n)^{-\{[(p+1)/2]+1\}} + \dots\dots\dots + d_1^{-1}d_1^{-2}\dots\dots d_1^{-k} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ Q_0(u)(d_k n)^{-[(p+1)/2]} + Q_1(u)(d_k n)^{-\{[(p+1)/2]+1\}} + \dots\dots\dots + d_k^{-1}d_k^{-2}\dots\dots d_k^{-k} \end{vmatrix}$$

$$= n^{-(k+1)}\{S(p,k,u) + O(1)\} \text{ for each fixed } u \in [0,\infty).$$

3. Direct Results

Theorem 1. Let $f \in L_p[0,\infty), p > 1$. If f has $(2k+2)$ derivatives on I_1 with $f^{(2k+1)} \in AC(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$ then all, n sufficiently large,

$$\left\|C_{(n)}^{(\alpha,\beta)}(f,k,.) - f\right\|_{L_p(I_2)} \leq \frac{D}{n^{(k+1)}}\left(\|f^{(2k+2)}\|_{L_p(I_1)} + \|f\|_{L_p[0,\infty)}\right)$$

Where D is constant independent of f and n.

Proof. In view of [6], let $u \in I_2$ and $t \in I_1$,

$$f\left(\frac{nt+\alpha}{n+\beta}\right) = \sum_{j=0}^{2k+1}\frac{\left(\frac{nt+\alpha}{n+\beta} - u\right)^j}{j!}f^{(j)}(u) + \frac{1}{(2k+1)!}\int_u^t\left(\frac{nt+\alpha}{n+\beta} - \omega\right)^{2k+1}f_{(\omega)}^{(2k+2)} + F\left(\frac{nt+\alpha}{n+\beta}, u\right)\left(1 - \varphi\left(\frac{nt+\alpha}{n+\beta}\right)\right) \tag{3.1}$$

Where $\varphi(t)$ is characterstic function of I_1 and,

$$F\left(\frac{nt+\alpha}{n+\beta},u\right)=f\left(\frac{nt+\alpha}{n+\beta}\right)-\sum_{j=0}^{2k+1}\frac{\left(\frac{nt+\alpha}{n+\beta}-u\right)^{j}}{j!}f^{j}(u)$$

For all $t \in [0,\infty)$ and $u \in I_2$.
Operating on this equality (3.1) by$C_{(n)}^{(\alpha,\beta)}(\cdot,k,u)$ and splitting right hand side into three terms, say, Ψ_1,Ψ_2,Ψ_3, we have,

$$\|\Psi_1\|_{L_p(I_2)} \le \frac{A_1}{n^{(k+1)}}\left(\sum_{j=1}^{2k+1}\left\|f^{(j)}\right\|_{L_p(I_2)}\right)$$

$$\le \frac{A_2}{n^{(k+1)}}\left(\|f\|_{L_p(I_2)}+\left\|f^{(2k+2)}\right\|_{L_p(I_2)}\right)$$

In view of lemma 2 and [7], let h_f be Hardy Little wood majorant [8] of $f^{(2k+2)}$ on I_1.
Now by Holder's inequality and (2.3),

$$K_1 \equiv \left|C_{(n)}^{(\alpha,\beta)}\left\{\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\int_u^t\left(\frac{nt+\alpha}{n+\beta}-\omega\right)^{2k+1}f_{(\omega)}^{(2k+1)}d\,\omega,u\right\}\right|$$

$$\le C_{(n)}^{(\alpha,\beta)}\left\{\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+1}\left|\int_u^t f_{(\omega)}^{(2k+2)}d\omega\right|,u\right\}$$

$$\le C_{(n)}^{(\alpha,\beta)}\left\{\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+2}\left|h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|,u\right\}$$

$$\le \left\{C_{(n)}^{(\alpha,\beta)}\left(\left|\frac{nt+\alpha}{n+\beta}-u\right|^{(2k+2)q}\right)\varphi\left(\frac{nt+\alpha}{n+\beta}\right),u\right\}^{1/q}$$

$$\left\{C_{(n)}^{(\alpha,\beta)}\left(\left|h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^{p}\varphi\left(\frac{nt+\alpha}{n+\beta}\right),u\right)\right\}^{1/p}$$

$$\le \frac{A_3}{n^{(k+1)}}\left\{\int_{a_1}^{b_1}M(n,u,t)\left|h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^{p}dt\right\}^{1/p}$$

Using Fubini's Theorem and [9],

$$\|K_1\|_{L_p(I_2)}^{p} \le \frac{A_3}{n^{(k+1)p}}\int_{a_2}^{b_2}\int_{a_1}^{b_1}M(n,u,t)\left|h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^{p}dtdu$$

$$\le \frac{A_3}{n^{(k+1)p}}\int_{a_1}^{b_1}\left[\int_{a_2}^{b_2}M(n,u,t)\right]\left|h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^{p}dt$$

$$\le \frac{A_3}{n^{(k+1)p}}\int_{a_1}^{b_1}\left|h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^{p}dt$$

$$\le \frac{A_3}{n^{(k+1)p}}\left\|h_f\left(\frac{nt+\alpha}{n+\beta}\right)\right\|_{L_p(I_1)}^{p}$$

Hence, $\|K_1\|_{L_p(I_2)} \le \frac{A_4}{n^{(k+1)}}\left\|f^{(2k+2)}\right\|_{L_p(I_1)}^{p}$

So, $\|\Psi_2\|_{L_p(I_2)} \leq \frac{A_5}{n^{(k+1)}}\left\|f^{(2k+2)}\right\|_{L_p(I_1)}$

For, $t \in [0,\infty)\backslash[a_1,b_1], u \in I_2, \exists\ \delta > 0,$ such that $|t-u| \leq \delta$.

Thus, $\left|C_{(n)}^{(\alpha,\beta)}\left(F\left(\frac{nt+\alpha}{n+\beta},u\right)\left(1-\emptyset\left(\frac{nt+\alpha}{n+\beta}\right),u\right)\right)\right| \leq \delta^{-(2k+2)}C_{(n)}^{(\alpha,\beta)}\left(\left|F\left(\frac{nt+\alpha}{n+\beta},u\right)\right|\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+2},u\right)$

$$\leq \delta^{-(2k+2)}C_{(n)}^{(\alpha,\beta)}\left[\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right| + \sum_{j=0}^{2k+1}\frac{\left|\frac{nt+\alpha}{n+\beta}-u\right|^j}{j!}\left|f^{(j)}(u)\right|\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+2},u\right]$$

$$\leq \delta^{-(2k+2)}C_{(n)}^{(\alpha,\beta)}\left(\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+2},u\right) + \sum_{j=1}^{2k+1}\frac{\left|f^{(j)}(u)\right|}{j!}C_{(n)}^{(\alpha,\beta)}\left[\left|\left(\frac{nt+\alpha}{n+\beta}-u\right)\right|^{2k+2+j},u\right]$$

$$=K_2 + K_3 \text{ (say)}$$

Using Holder's inequality and (2.3),

$$|K_2| \leq \frac{A_6}{n^{(k+1)}}\left(C_{(n)}^{(\alpha,\beta)}\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|^p,u\right)^{1/p}$$

Now, reapplying Fubini's Theorem and [10],

$$\|K_2\|_{L_p(I_2)} \leq \frac{A_7}{n^{(k+1)}}\|f\|_{L_p[0,\infty)}$$

Using equation (2.3) and [7] [11],

$$\|K_3\|_{L_p(I_2)} \leq \frac{A_8}{n^{(k+1)}}\left(\|f\|_{L_p(I_2)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)}\right)$$

Therefore,

$$\|\Psi_3\|_{L_p(I_2)} \leq \frac{A_9}{n^{(k+1)}}\left(\|f\|_{L_p[0,\infty)} + \left\|f^{(2k+2)}\right\|_{L_p(I_2)}\right)$$

The result follows.

Theorem 2. Let $f \in L_1[0,\infty)$. If f has $(2k+1)$ derivatives on I_1 with $f^{(2k)} \in AC(I_1)$ and $f^{(2k+1)} \in B.V.(I_1)$, then for all n sufficiently large

$$\left\|C_{(n)}^{(\alpha,\beta)}(f,k,\cdot) - f\right\|_{L_1(I_2)} \leq \frac{D}{n^{(k+1)}}\left\{\left\|f^{(2k+1)}\right\|_{B.V.(I_1)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)}\right\}$$

Where D is a constant independent of f and n.

Proof. For our assumption of f, and for almost all $u \in I_2$ and for all $t \in I_1$, we have,

$$f\left(\frac{nt+\alpha}{n+\beta}\right) = \sum_{i=0}^{2k+1}\frac{\left(\frac{nt+\alpha}{n+\beta}-u\right)^i}{i!}f^{(i)}(u) + \frac{1}{(2k+1)!}\int_u^t\left(\frac{nt+\alpha}{n+\beta}-\omega\right)^{2k+1}df^{(2k+1)}(\omega)\,\varphi\left(\frac{nt+\alpha}{n+\beta}\right)$$

$$+F\left(\frac{nt+\alpha}{n+\beta},u\right)\left(1-\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\right)$$

Where $\varphi(t)$ is characterstic function of I_1.

$$F\left(\frac{nt+\alpha}{n+\beta},u\right) = f\left(\frac{nt+\alpha}{n+\beta}\right) - \sum_{i=0}^{2k+1}\frac{\left(\frac{nt+\alpha}{n+\beta}-u\right)^i}{i!}f^i(u)$$

for almost all $u \in I_2$ and $t \in [0,\infty)$.

Thus,

$$C_{(n)}^{(\alpha,\beta)}(f,k,u)-f(u)=\sum_{i=1}^{2k+1}\frac{f^{(i)}(u)}{i!}C_{(n)}^{(\alpha,\beta)}\left(\left(\frac{nt+\alpha}{n+\beta}-u\right)^{i},k,u\right)$$

$$+\frac{1}{(2k+1)!}C_{(n)}^{(\alpha,\beta)}\left(\int_u^t\left(\frac{nt+\alpha}{n+\beta}-\omega\right)^{2k+1}df^{(2k+1)}(\omega)\,\varphi\left(\frac{nt+\alpha}{n+\beta}\right),k,u\right)$$

$$+\ C_{(n)}^{(\alpha,\beta)}\left(F\left(\frac{nt+\alpha}{n+\beta},u\right)\left(1-\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\right),k,u\right)$$

$$=K_1+K_2+K_3 \qquad \text{(say)}$$

Applying lemma 2 and [7] [12],

$$\|K_1\|_{L_1(I_2)}\le\frac{B_1}{n^{(k+1)}}\left(\|f\|_{L_1(I_2)}+\left\|f^{(2k+2)}\right\|_{L_1(I_2)}\right)$$

Further,

$$H\equiv\left\|C_{(n)}^{(\alpha,\beta)}\left(\int_u^t\left(\frac{nt+\alpha}{n+\beta}-\omega\right)^{2k+1}df^{(2k+1)}(\omega)\varphi\left(\frac{nt+\alpha}{n+\beta}\right),u\right)\right\|_{L_1(I_2)}$$

$$\le\int_{a_2}^{b_2}\int_{a_1}^{b_1}M(n,u,t)\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+1}\left|\int_u^t\left|df^{(2k+1)}(\omega)\right|\right|dtdu$$

For each n there exists a non-negative integer $r=r(n)$ such that

$$\frac{r}{\sqrt{n}}\le\max(b_1-a_2,b_2-a_1)\le\frac{(r+1)}{\sqrt{n}}$$

Then, we have,

$$H\le\sum_{l=0}^{r}\int_{a_2}^{b_2}\left\{\int_{u+(l/\sqrt{n})}^{u+\{(l+1)/\sqrt{n}\}}\varphi\left(\frac{nt+\alpha}{n+\beta}\right)M(n,t,u)\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+1}\cdot\left(\int_u^{u+\{(l+1)/\sqrt{n}\}}\varphi(\omega)\left|df^{(2k+1)}(\omega)\right|\right)dt\right.$$

$$\left.+\int_{u-\{(l+1)/\sqrt{n}\}}^{u-(l/\sqrt{n})}\varphi\left(\frac{nt+\alpha}{n+\beta}\right)M(n,u,t)\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+1}\cdot\left(\int_{u-\{(l+1)/\sqrt{n}\}}^{u}\varphi(\omega)\left|df^{(2k+1)}(\omega)\right|\right)\right\}$$

Let $\varphi_{(u,c,d)}(\omega)$ be characterstic function of the interval $\left[u-\left(c/\sqrt{n}\right),u+\left(d/\sqrt{n}\right)\right]$, where c,d are non-negative integers. Hence, we get,

$$H\le\sum\nolimits_{l=1}^{r}\int_{a_2}^{b_2}\left\{\int_{u+(l/\sqrt{n})}^{u+\{(l+1)/\sqrt{n}\}}\varphi\left(\frac{nt+\alpha}{n+\beta}\right)M(n,u,t)\frac{n^2}{l^4}\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+5}\right.$$

$$\times\left(\int_u^{u+\{(l+1)/\sqrt{n}\}}\varphi(\omega)\varphi_{(u,0,l+1)}(\omega)\left|df^{(2k+1)}(\omega)\right|\right)dt$$

$$+\int_{u-\{(l+1)/\sqrt{n}\}}^{u-(l/\sqrt{n})}\varphi\left(\frac{nt+\alpha}{n+\beta}\right)M(n,u,t)\frac{n^2}{l^4}\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+5}$$

$$\left.\times\left(\int_{u-\{(l+1)/\sqrt{n}\}}^{u}\varphi(\omega)\varphi_{(u,l+1,0)}(\omega)\left|df^{(2k+1)}(\omega)\right|\right)dt\right\}du$$

$$+\int_{a_2}^{b_2}\int_{-1/\sqrt{n}}^{a_1+(1/\sqrt{n})}\varphi\left(\frac{nt+\alpha}{n+\beta}\right)M(n,t,u)\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+1}$$

$$\times\left(\int_{u-(1/\sqrt{n})}^{u+(1/\sqrt{n})}\varphi(\omega)\varphi_{(u,1,1)}(\omega)\left|df^{(2k+1)}(\omega)\right|\right)dtdu$$

$$\leq$$

$$\sum_{l=1}^{r}\left\{\frac{n^2}{l^4}\int_{a_2}^{b_2}\left(\int_{u+(l/\sqrt{n})}^{u+\{(l+1)/\sqrt{n}\}}\varphi\left(\frac{nt+\alpha}{n+\beta}\right)M(n,t,u)\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+5}\right)\times\left(\int_{a_1}^{b_1}\varphi_{(u0,l+1)}(\omega)\left|df^{(2k+1)}(\omega)\right|\right)dt+\int_{u-\{(l+1)/\sqrt{n}\}}^{u-(l/\sqrt{n})}\varphi\left(\frac{nt+\alpha}{n+\beta}\right)M(n,t,u)\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+5}\times\left(\int_{a_1}^{b_1}\varphi_{(u,l+1,0)}(\omega)\left|df^{(2k+1)}(\omega)\right|dt\right)du\right\}+\int_{a_2}^{b_2}\int_{-1/\sqrt{n}}^{a_1+(1/\sqrt{n})}\varphi\left(\frac{nt+\alpha}{n+\beta}\right)M(n,t,u)\left|\frac{nt+\alpha}{n+\beta}-u\right|^{2k+1}\times\left(\int_{a_1}^{b_1}\varphi_{(u,1,1)}(\omega)\left|df^{(2k+1)}(\omega)\right|\right)dtdu$$

Using lemma 1and Fubini's theorem in the next step to obtain,

$$=\frac{A_2}{n^{(2k+1)/2}}\left\{\sum_{l=1}^{r}\frac{1}{l^4}\left(\int_{a_1}^{b_1}\left(\int_{b_2}^{a_2}\varphi_{(u,0,l+1)}(\omega)du\right)\left|df^{(2k+1)}(\omega)\right|+\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\varphi_{(u,l+1,0)}(\omega)\,du\right)\left|df^{(2k+1)}(\omega)\right|+\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\varphi_{(u,1,1)}(\omega)\right)\left|df^{(2k+1)}(\omega)\right|\right)\right\}$$

$$\leq\frac{A_3}{n^{(2k+1)/2}}\left\{\sum_{l=1}^{r}\left(\int_{a_1}^{b_1}\left(\int_{\omega-\{(l+1)/\sqrt{n}\}}^{\omega}du\right)\left|df^{(2k+1)}(\omega)\right|+\int_{a_1}^{b_1}\left(\int_{\omega}^{\omega+\{(l+1)/\sqrt{n}\}}du\right)\left|df^{(2k+1)}(\omega)\right|\right)+\int_{a_1}^{b_1}\left(\int_{\omega-(1/\sqrt{n})}^{\omega+(1/\sqrt{n})}du\right)\left|df^{(2k+1)}(\omega)\right|\right\}$$

$$\leq\frac{A_4}{n^{(k+1)}}\left\|f^{(2k+1)}\right\|_{B.V(I_1)}$$

Hence, $\|K_2\|_{L_1(I_2)}\leq\frac{A_5}{n^{(k+1)}}\left\|f^{(2k+1)}\right\|_{B.V(I_1)}$

Here, A_5 depends on k.

For all, $t\in[0,\infty)\backslash[a_1,b_1]$ and all $u\in I_2$, we can choose a $\delta>0$ such that $|t-u|\geq\delta$,

$$\left\|C_{(n)}^{(\alpha,\beta)}\left(\frac{nt+\alpha}{n+\beta},u\right)\left(1-\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\right)u\right\|_{L_1(I_2)}$$
$$\leq\int_{a_2}^{b_2}\int_0^{\infty}M(n,u,t)\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|\left(1-\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\right)dtdu+\sum_{i=0}^{2k+1}\frac{1}{i!}\int_{a_2}^{b_2}\int_0^{\infty}M(n,u,t)\left|f^{(i)}(t)\right|\times\left|\frac{nt+\alpha}{n+\beta}-u\right|^{i}\left(1-\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\right)dtdu$$

$$=K_4+K_5\ \text{(say)}$$

For sufficiently large t, there exists constants D_o and A_6 such that,

$\frac{\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+2}}{\left(\frac{nt+\alpha}{n+\beta}\right)^{2k+2}+1}\geq A_6$ for all $t\geq D_0$ and $u\in I_2$.

By Fubini's Theorem,

$$K_4=\left(\int_0^{D_0}\int_{a_2}^{b_2}+\int_{D_0}^{\infty}\int_{a_2}^{b_2}\right)M(n,u,t)\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|\left|\left(1-\varphi\left(\frac{nt+\alpha}{n+\beta}\right)\right)\right|dudt=K_6+K_7\ \text{(say)}$$

Now, using lemma 1,

$$K_6\leq\frac{A_7}{n^{(k+1)}}\left(\int_0^{a_0}\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|dt\right)$$

And

$$K_7\leq\frac{1}{A_6}\int_{D_0}^{\infty}\int_{a_2}^{b_2}M(n,u,t)\frac{\left(\frac{nt+\alpha}{n+\beta}-u\right)^{2k+2}}{\left\{\left(\frac{nt+\alpha}{n+\beta}\right)^{2k+2}+1\right\}}\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|dudt$$

$$\leq\frac{A_8}{n^{(k+1)}}\left(\int_{D_0}^{\infty}\left|f\left(\frac{nt+\alpha}{n+\beta}\right)\right|dt\right)$$

Hence, $K_4\leq\frac{A_9}{n^{(k+1)}}\|f\|_{L_1[0,\infty)}$

Further using (2.3) and [7],

$$K_5\leq\frac{A_{10}}{n^{(k+1)}}\left(\|f\|_{L_1(I_2)}+\left\|f^{(2k+1)}\right\|_{L_1(I_2)}\right)$$

The above estimates of K_4 and K_5 leads to,

$$\|K_3\|_{L_1(I_2)} \leq \frac{A_{11}}{n^{(k+1)}}\left(\|f\|_{L_1[0,\infty)} + \left\|f^{(2k+1)}\right\|_{L_1(I_2)}\right)$$

By combining estimates of K_1, K_2 and K_3, we get the required result.

4. Conclusion

Using technique of linear approximation method, we get direct theorem for linear combination of Stancu generalized modified Lupas operators. Similarly, we can also get local inverse and saturation results.

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