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RBFs for Integral Equations with a Weakly Singular Kernel

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Abstract: In this paper a numerical method, based on collocation method and radial basis functions (RBF) is proposed for solving integral equations with a weakly singular kernel. Integrals appeared in the procedure of the solution are approximated by adaptive Lobatto quadrature rule. Illustrative examples are included to demonstrate the validity and applicability of the presented technique. In addition, the results of applying the method are compared with those of Homotopy perturbation, and Adomian decomposition methods.

Keywords: Integral Equations with a Weakly Singular Kernels, Radial Basis Functions (RBF), Adaptive Lobatto Quadrature, Homotopy Perturbation Method (HPM), Adomian Decomposition Method (ADM)

1. Introduction

Weakly singular integral equations have a major role in the fields of science and engineering. They appear in mathematical modeling of different phenomena in many disciplines such as: physics, chemistry, biology, and others. It is difficult to solve these equations analytically, hence numerical solutions are required. Singular integral equations have been approached by different methods including Collocation method [2-4], Reproducing kernel method [17], Galerkin method [5], Adomian decomposition method [1], Homotopy perturbation method [6], Radial Basis Functions [10, 11], Newton product integration method [7], and many others.

In recent years, meshless methods are used as a class of numerical methods for solving functional equations. Meshless methods just use a scattered set of collocation points, regardless any relationship information between the collocation points. This property is the main advantage of these techniques over the mesh dependent methods such as finite difference methods, finite element methods. A well-known family of meshless methods is the method of radial basis functions.

Since 1990, radial basis function method [8] is used as a meshless method to approximate the solutions of partial differential equations [9, 12-16]. These methods are developed for solving various types of linear and nonlinear functional equations such as Ordinary differential equations (ODEs). Integral and Integro-differential equations (IEs and IDEs) are solved with the RBF method by some researchers [19-21]. In [22-24] authors used the RBF method to solve some engineering problems. Also authors of [25] solved fractional diffusion equations by RBFs.

In this paper, we will use an efficient method based on radial basis functions and collocation method to solve integral equations with a weakly singular kernel. The paper is organized as follows. In Section 2, the radial basis functions are introduced. Section 3, as the main part, presents the solution of weakly singular integral equations by RBF, via collocation method. Numerical illustrative examples are included in Section 4. A conclusion is drawn in Section 5.

2. Radial Basis Functions

Let's define the main features of the method.

2.1. Definition of Radial Basis Functions

Radial basis functions usually approximate a function as the following

$$s(x)=\sum_{i=1}^{N}\lambda_i\phi(\|x-x_i\|),\quad x\in\mathbb{R}.$$

Where $\phi:[0,\infty)\to\mathbb{R}$ is a fixed univariate function, the coefficients $(\lambda_i)_{i=1}^{N}$ are real numbers and $\|\cdot\|$, denotes the Euclidean norm.

2.2. Radial Basis Functions Interpolation

The radial basis functions approximation of a real function, say $u(x)$, is given by

$$u(x) \approx u_N(x) = \sum_{i=1}^{N} \lambda_i \phi(\| x - x_i \|) = \sum_{i=1}^{N} \lambda_i \phi_i(x) = \mathbf{\Phi}^T(x)\mathbf{\Lambda},$$

where

$$\phi_i(x) = \phi(\| x - x_i \|),$$

$$\mathbf{\Phi}^T(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_N(x)],$$

$$\mathbf{\Lambda} = [\lambda_1, \lambda_2, \ldots \lambda_N]^T,$$

and x_i, $i = 1, 2, \ldots, N$ is a finite number of distinct points (centers) in $\mathbb{R}$. Consider N distinct support points $(x_j, u(x_j))$, $j = 1, 2, \ldots, N$. One can find λ_i s by solving the following linear system

$$\mathbf{A\Lambda} = \mathbf{u}.$$

Where

$$\mathbf{A} = [\phi(\| x_j - x_i \|)]_{i,j=1}^{N},$$

$$\mathbf{\Lambda} = [\lambda_1, \lambda_2, \ldots \lambda_N]^T, \text{ and } \mathbf{u} = [u_1, u_2, \ldots u_N]^T.$$

Some well-known RBFs are listed in Table 1, where the Euclidian distance r is real and non-negative, and c is a positive scalar, called shape parameter.

Table 1. *Some well-known RBFs.*

Name of the function	Definition
Gaussian	$\phi(r) = e^{-cr^2}$
Inverse Quadric	$\phi(r) = \frac{1}{r^2 + c^2}$
Hardy Multiquadric	$\phi(r) = \sqrt{r^2 + c^2}$
Inverse Multiquadric	$\phi(r) = \frac{1}{\sqrt{r^2 + c^2}}$
Cubic	$\phi(r) = r^3$
Thin plate splines	$\phi(r) = r^2 \log(r)$
Hyperbolic secant	$\phi(r) = sech(cr)$

3. Application of RBF Method

In this paper Radial Basis Functions are used to approximate solution of integral equations with a weakly singular kernel in the following general form

$$y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \ 0 \le x \le 1, \tag{1}$$

where $f(x)$ is a known analytic function defined on the interval $0 \le x \le 1$. In order to use the radial basis functions, let's consider $y(x)$ as follows,

$$y(x) \approx y_N(x) = \sum_{i=1}^{N} \lambda_i \phi_i(x) = \mathbf{\Phi}^T(x)\mathbf{\Lambda}. \tag{2}$$

Substitution in (1) from (2) leads to

$$\mathbf{\Phi}^T(x)\mathbf{\Lambda} = f(x) + \int_0^x \frac{\mathbf{\Phi}^T(x)}{\sqrt{x-t}} \mathbf{\Lambda} dt, \tag{3}$$

To determine λ_i, $i = 1, 2, \ldots, N$, from Eq. (3), let's use shifted zeros of the Legendre polynomials, $x = x_j$, $j = 1, 2, \ldots, N$, as the collocation points,

$$\left(\mathbf{\Phi}^T(x_j) - \int_0^{x_j} \frac{\mathbf{\Phi}^T(t)}{\sqrt{x_j - t}} dt \right) \mathbf{\Lambda} = f(x_j). \tag{4}$$

By using the Adaptive Lobatto Quadrature [18], we can approximate the integral in Eq. (4) to achieve a set of N linear equations for determining components of $\mathbf{\Lambda}$.

4. Numerical Examples

In this section, two examples are provided to illustrate the efficiency of this approach. For the sake of comparing purposes, the same examples as [6], that have used Homotopy perturbation method, are considered.

4.1 Example

Let us consider the following weakly singular integral equation of the second kind

$$y(x) = x^2 + \frac{16}{15} x^{\frac{5}{2}} - \int_0^x \frac{y(t)}{\sqrt{x-t}} dt, \ 0 \le x \le 1, \tag{5}$$

with the exact solution $y(x) = x^2$.

Let's approximate $y(x)$ as follows

$$y(x) \approx y_5(x) = \sum_{i=1}^{5} \lambda_i \mathrm{e}^{\frac{|x - x_i|^2}{100}}, \tag{6}$$

where

$$x_1 = 0.0469, \ x_2 = 0.2308,$$
$$x_3 = 0.5000, \ x_4 = 0.7692, \ x_5 = 0.9531,$$

are shifted zeros of Legendre polynomial of degree 5. Substituting (6) into (5) gives

$$\sum_{i=1}^{5}\lambda_i \mathrm{e}^{-\frac{|x-x_i|^2}{100}} = x^2+\frac{16}{15}x^{\frac{5}{2}} - \sum_{i=1}^{5}\lambda_i \int_0^x \frac{\mathrm{e}^{-\frac{|t-x_i|^2}{100}}}{\sqrt{x-t}}dt,$$

or

$$\sum_{i=1}^{5}\left(\mathrm{e}^{-\frac{|x-x_i|^2}{100}} + \int_0^x \frac{\mathrm{e}^{-\frac{|t-x_i|^2}{100}}}{\sqrt{x-t}}dt\right)\lambda_i = x^2+\frac{16}{15}x^{\frac{5}{2}}.$$

By using $x = x_j, j = 1,\ldots,5$ as collocation points, we have

$$\sum_{i=1}^{5}\left(\mathrm{e}^{-\frac{|x_j-x_i|^2}{100}} + \int_0^{x_j} \frac{\mathrm{e}^{-\frac{|t-x_i|^2}{100}}}{\sqrt{x_j-t}}dt\right)\lambda_i = x_j^2+\frac{16}{15}x_j^{\frac{5}{2}}. \quad (7)$$

Using adaptive Lobatto quadrature, integrals in Eq. (7) is computed and Eq. (7) is reduced to the following equation

$$\mathbf{A\Lambda = F} \quad (8)$$

where

$$\mathbf{A} = \begin{bmatrix} 1.4332 & 1.4327 & 1.4302 & 1.4256 & 1.4213 \\ 1.9603 & 1.9607 & 1.9588 & 1.9542 & 1.9494 \\ 2.4107 & 2.4130 & 2.4135 & 2.4105 & 2.4064 \\ 2.7442 & 2.7489 & 2.7525 & 2.7520 & 2.7495 \\ 2.9360 & 2.9426 & 2.9485 & 2.9503 & 2.9490 \end{bmatrix},$$

$$\mathbf{F} = \left[x_j^2+\frac{16}{15}x_j^{\frac{5}{2}}\right]_{j=1}^{5} = \begin{bmatrix} 0.0027 \\ 0.0805 \\ 0.4386 \\ 1.1453 \\ 1.8543 \end{bmatrix},$$

and $\mathbf{\Lambda} = \left[\lambda_1, \lambda_2, \ldots, \lambda_5\right]^T$ is determined as the following

$$\mathbf{\Lambda} = \begin{bmatrix} 2.7558e+07 \\ -7.7979e+07 \\ 1.0093e+08 \\ -7.8189e+07 \\ 2.7683e+07 \end{bmatrix}. \quad (9)$$

Substituting from (9) into (6) yields an approximate solution as the following

$$y_5(x) = \left(2.7558\mathrm{e}^{-\frac{|x-0.0469|^2}{100}} - 7.7979\mathrm{e}^{-\frac{|x-0.2308|^2}{100}} + 10.093\mathrm{e}^{-\frac{|x-0.5000|^2}{100}} - 7.8189\mathrm{e}^{-\frac{|x-0.7692|^2}{100}} + 2.7683\mathrm{e}^{-\frac{|x-0.9531|^2}{100}}\right)\times 10^7,$$

with the maximum absolute error $7.9107e-08$.

More accurate approximations to the solution may be attained by considering additional terms. The Errors of the numerical solutions for $N = 10$, and different RBFs are shown in the Table 2. Results in the Table 2 shows that all solutions which are achieved by Gaussian ($\phi(r) = \exp(-r^2/100)$), Inverse Quadric ($\phi(r) = 1/(1+r^2)$), and Multiquadric ($\phi(r) = \sqrt{1+r^2}$) RBFs are accurate enough for knowing this approach as a powerful device. Also the absolute error for this approach and Homotopy perturbation method [6] are shown in Figure 2. A Comparison which is made in Figure 2, shows that RBF method with Gaussian bases, gives more accurate solutions than the homotopy perturbation method by 18 terms. As x increases, the Error of HPM is increased rapidly, but RBF Errors are not significant yet.

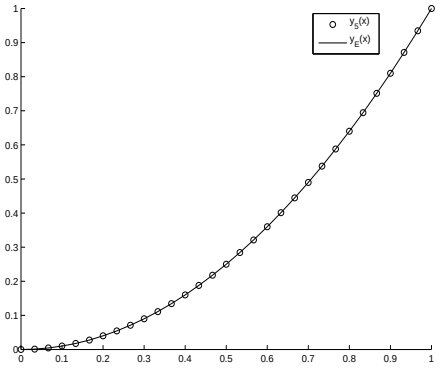

Figure 1. *Exact solution, $y_E(x)$, and approximate solution, $y_5(x)$, for Example 1.*

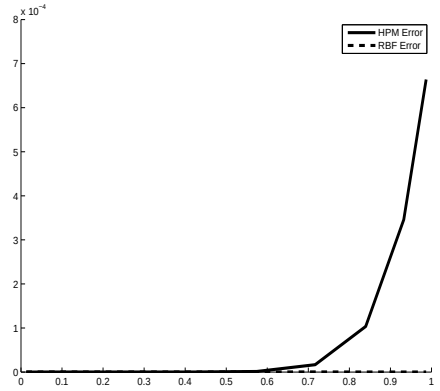

Figure 2. *The absolute Error of HPM, by 18 terms, and RBF, for $\phi(r) = \mathrm{e}^{-(0.1r)^2}$.*

4.2. Example

Now consider the following weakly singular integral equation

$$y(x)=\sqrt{x}+\frac{\pi}{2}x-\int_0^x\frac{y(t)}{\sqrt{x-t}}dt,\ 0\le x\le 1. \quad (10)$$

The exact solution is $y(x)=\sqrt{x}$.

First, $y(x)$ is approximated by

$$y_8(x)=\sum_{i=1}^{8}\frac{\lambda_i}{1+(x-x_i)^2}, \quad (11)$$

where

$$x_1=0.0199,\ x_2=0.1017,\ x_3=0.2372,\ x_4=0.4083,$$
$$x_5=0.5917,\ x_6=0.7628,\ x_7=0.8983,\ x_8=0.9801,$$

are shifted zeros of Legendre polynomials of degree 8. Substituting (11) into (10) results in

$$\sum_{i=1}^{8}\frac{\lambda_i}{1+(x-x_i)^2}=\sqrt{x}+\frac{\pi}{2}x-\sum_{i=1}^{8}\lambda_i\int_0^x\frac{\frac{1}{1+(t-x_i)^2}}{\sqrt{x-t}}dt,$$

or

$$\sum_{i=1}^{8}\left(\frac{1}{1+(x-x_i)^2}+\int_0^x\frac{dt}{\left(1+(t-x_i)^2\right)\sqrt{x-t}}\right)\lambda_i=\sqrt{x}+\frac{\pi}{2}x. \quad (12)$$

By using $x=x_j, j=1,\ldots,8$ as collocation points, we have

$$\sum_{i=1}^{8}\left(\frac{1}{1+(x_j-x_i)^2}+\int_0^{x_j}\frac{dt}{\left(1+(t-x_i)^2\right)\sqrt{x_j-t}}\right)\lambda_i=\sqrt{x_j}+\frac{\pi}{2}x_j. \quad (13)$$

Integrals in Eq. (13) is computed by adaptive Lobatto quadrature, and Eq. (13) is reduced to matrix equation

$$\mathbf{A\Lambda}=\mathbf{F}, \quad (14)$$

where

$$\mathbf{F}=\left[\sqrt{x_j}+\frac{\pi}{2}x_j\right]_{j=1}^{8}=\begin{bmatrix}0.1721\\0.4786\\0.8597\\1.2803\\1.6987\\2.0715\\2.3589\\2.5296\end{bmatrix},$$

$$\mathbf{A}=\begin{bmatrix}1.2818&1.2730&1.2232&1.1127&0.9647&0.8248&0.7224&0.6659\\1.6290&1.6364&1.6014&1.4852&1.3067&1.1260&0.9893&0.9126\\1.9063&1.9482&1.9635&1.8852&1.7072&1.4972&1.3261&1.2268\\2.0568&2.1396&2.2295&2.2389&2.1183&1.9163&1.7260&1.6079\\2.0799&2.1924&2.3480&2.4609&2.4477&2.3103&2.1373&2.0163\\2.0303&2.1558&2.3482&2.5366&2.6306&2.5907&2.4732&2.3722\\1.9662&2.0946&2.3004&2.5267&2.6904&2.7334&2.6790&2.6094\\1.9222&2.0502&2.2591&2.4994&2.6951&2.7834&2.7700&2.7239\end{bmatrix},$$

and $\mathbf{\Lambda}=[\lambda_1,\lambda_2,\ldots,\lambda_8]^T$. System (14) is solved to determine components of $\mathbf{\Lambda}$ as below

$$\mathbf{\Lambda}=\begin{bmatrix}-687.02\\1608.32\\-1719.90\\1446.34\\-1225.73\\1087.86\\-831.64\\321.35\end{bmatrix}. \quad (15)$$

Substituting from (15) into (11) gives an approximate solution as the following

$$\begin{aligned}y_8(x)=&-\frac{687.02}{1+(x-0.0199)^2}+\frac{1608.32}{1+(x-0.1017)^2}\\&-\frac{1719.90}{1+(x-0.2372)^2}+\frac{1446.34}{1+(x-0.4083)^2}\\&-\frac{1225.73}{1+(x-0.5917)^2}+\frac{1087.86}{1+(x-0.7628)^2}\\&-\frac{831.64}{1+(x-0.8983)^2}+\frac{321.35}{1+(x-0.9801)^2},\end{aligned}$$

with the maximum error $2.4342e-03$.

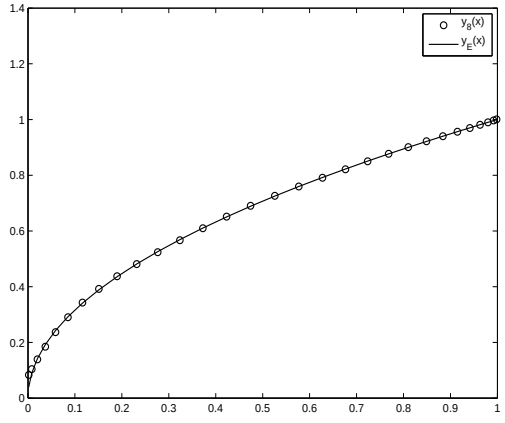

Figure 3. *Exact solution, $y_E(x)$, and approximate solution, $y_8(x)$, for Example 2.*

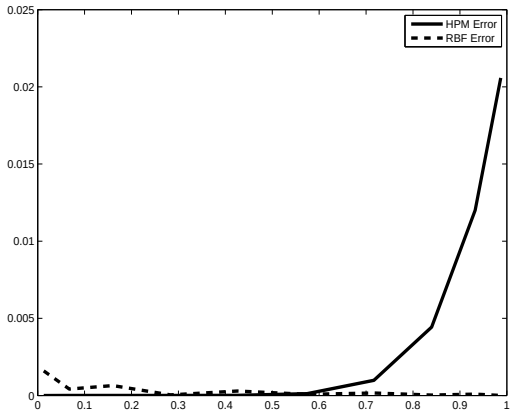

Figure 4. *The absolute Error of HPM, by 18 terms, and RBF, for* $\phi(r)=\sqrt{1+r^2}$.

Errors of the numerical solutions for $N=10$, and Multiquadric ($\phi(r)=\sqrt{1+r^2}$), Inverse Quadric ($\phi(r)=1/(1+r^2)$), and Inverse Multiquadric ($\phi(r)=1/\sqrt{1+r^2}$) RBFs are shown in the Table 3. These results show that all solutions which are achieved by these different RBFs are accurate enough for knowing RBF method as a powerful device. The absolute error for this approach and HPM [6] are shown in Figure 4. A Comparison which is made in Figure 4, shows that RBF method with Multiquadric bases gives more accurate solutions than the homotopy perturbation method by 18 terms. As x increases, the Error of HPM is increased rapidly, but RBF Error vanishes.

5. Conclusion

This paper describes using Radial basis functions to solve integral equations, with a weakly singular kernel. The proposed method, is a meshless method, just uses a scattered set of collocation points, without any connectivity information between the collocation points. Two examples are presented to show the efficiency of the method. Numerical results at scattered points appeared in Tables 2 and 3. Exact and approximate solutions for two examples are plotted in Figures 1 and 3. Figures confirms that errors is not significant and solutions are satisfactory even by a few number of basis functions. The behavior of the errors for Homotopy perturbation method and Radial basis functions is apparent in Figures 2 and 4 for two examples. A comparison with Homotopy perturbation method shows that RBF results in more accurate solutions than Homotopy perturbation method or Adomian decomposition method which are reported in [6]. The results of two solved examples are accurate enough for knowing this approach as a powerful device. From physical point of view, these figures show that in the RBF method, as x increases, equilibrium appears.

Table 2. *Solutions and Errors for Example 1.*

x	y_{Exact} (y_E)	$\phi(r)=\exp(-r^2/100)$		$\phi(r)=1/(1+r^2)$		$\phi(r)=\sqrt{1+r^2}$	
		$y_{10}(x)$	$\lvert y_{10}-y_E\rvert$	$y_{10}(x)$	$\lvert y_{10}-y_E\rvert$	$y_{10}(x)$	$\lvert y_{10}-y_E\rvert$
0.0130	0.00017021	0.00017026	4.3354e-08	0.00017112	9.0317e-07	0.00017027	5.5240e-08
0.0675	0.00455197	0.00455194	2.6635e-08	0.00455079	1.1825e-06	0.00455193	3.4940e-08
0.1603	0.02569455	0.02569460	5.2464e-08	0.02569624	1.6871e-06	0.02569754	2.9851e-06
0.2833	0.08026019	0.08026009	9.6868e-08	0.08025590	4.2943e-06	0.08025913	1.0604e-06
0.4256	0.18110372	0.18110373	1.3451e-08	0.18110999	6.2725e-06	0.18110412	4.0251e-07
0.5744	0.32997806	0.32997792	1.3210e-07	0.32997235	5.7053e-06	0.32997729	7.6181e-07
0.7167	0.51365558	0.51365527	3.1387e-07	0.51366105	5.4612e-06	0.51365605	4.6475e-07
0.8397	0.70510412	0.70510411	1.1898e-08	0.70509947	4.6490e-06	0.70510354	5.7539e-07
0.9325	0.86961534	0.86961540	6.5357e-08	0.86961939	4.0569e-06	0.86961571	3.7189e-07
0.9870	0.97407674	0.97407674	2.0636e-09	0.97407264	4.0986e-06	0.97407624	4.9822e-07

Table 3. *Solutions and Errors for Example 2.*

x	y_{Exact} (y_E)	$\phi(r)=\sqrt{1+r^2}$		$\phi(r)=1/(1+r^2)$		$\phi(r)=1/\sqrt{1+r^2}$	
		$y_{10}(x)$	$\lvert y_{10}-y_E\rvert$	$y_{10}(x)$	$\lvert y_{10}-y_E\rvert$	$y_{10}(x)$	$\lvert y_{10}-y_E\rvert$
0.0130	0.11422230	0.11262579	1.5965e-03	0.11261085	1.6115e-03	0.11263024	1.5921e-03
0.0675	0.25974663	0.26016436	4.1772e-04	0.26017013	4.2349e-04	0.26014565	3.9901e-04
0.1603	0.40036884	0.40101248	6.4363e-04	0.39988836	4.8048e-04	0.39993226	4.3658e-04
0.2833	0.53226149	0.53222145	4.0049e-05	0.53259710	3.3560e-04	0.53297792	7.1643e-04
0.4256	0.65235176	0.65206011	2.9166e-04	0.65214515	2.0661e-04	0.65175952	5.9224e-04
0.5744	0.75791633	0.75799302	7.6694e-05	0.75810850	1.9217e-04	0.75816263	2.4630e-04
0.7167	0.84658000	0.84641812	1.6187e-04	0.84638985	1.9014e-04	0.84637489	2.0511e-04
0.8397	0.91635407	0.91638214	2.8074e-05	0.91651646	1.6240e-04	0.91644249	8.8420e-05
0.9325	0.96567680	0.96559109	8.5710e-05	0.96555392	1.2288e-04	0.96558939	8.7405e-05
0.9870	0.99345521	0.99346109	5.8756e-06	0.99335758	9.7631e-05	0.99349627	4.1062e-05

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