

Some Metric Properties and a Constructive Task of a Semi-Regular $2n$ -Sides Polygon

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Abstract: A simple polygon that either has equal all sides or all interior angles is called a semi-regular polygon. In terms of this definition, we can distinguish between two types of semi-regular polygons: equilateral polygons (that have equal all sides and different interior angles) and equiangular polygons (that have equal interior angles and different sides). To analyze the metric properties of semi-regular polygons, knowing only one basic element, e.g. the length of a side, as in regular polygons, is not enough. Therefore, in addition to the side of a semi-regular polygon, we use another characteristic element of it to analyze the metric features, and that is the angle $\delta = \angle(a, b)$ between the side of a semi-regular polygon \mathcal{P}_N and the side b of its inscribed regular polygon \mathcal{P}_n . Some metric properties of a semi-regular equilateral $2n$ -sides polygon are analyzed in this paper with respect to these two characteristic elements. Some of the problems discussed in the paper are: convexity, calculation of surface area, dependence on the length of sides a and δ , calculation of the radius of the inscribed circle depending on the sides a and angles δ , and calculation of the surface area in which the radius of the inscribed circle is known, as well as the relationship between them. It has been shown that the formula for calculating the surface area of regular polygons results from the formula for the surface area of $2n$ -side semi-regular, equilateral polygons. Further, by using these results, it has been shown that the cross-sections of regular polygons inscribed to semi-regular equilateral polygons, the vertices of equiangular semi-regular polygons, as well as the sides of the regular polygons inscribed to it, intersect in the same manner at the vertices of the equilateral semi-regular polygon. It has further been shown that the sides of the equiangular semi-regular polygon refer to each other as the sines of the angles created by the sides of the inscribed polygons and the side of the semi-regular polygon.

Keywords: Semi-Regular Polygons, Surface Ratio, Equilateral and Equiangular Semi-Regular Polygons

1. Introduction

A simple polygon $\mathcal{P}_N \equiv A_1A_2 \dots A_N$ that has equal all sides or equal all interior angles is called a semi-regular polygon [8, 9, 10, 11]. In terms of definition, we distinguish between two types of semi-regular polygons: equiangular (having equal interior angles and different sides) and equilateral (having equal sides, and different interior angles). We consider that vertex $A_i, i = 1, 2, \dots, N, i \in \mathbb{N}$ of a polygon is in an even position, or odd position, if index i is even, or odd number, respectively. In this paper we consider convex equilateral semi-regular polygons. The marking is as follows:

1. N is a number of sides in a semi-regular equilateral polygon,

2. n is a number of sides in a regular polygon,

3. a is a side in a semi-regular polygon \mathcal{P}_N ,

4. $b_j, j \in \mathbb{N}$ side in a regular polygon \mathcal{P}_n^i "inscribed" to a semi-regular polygon $\mathcal{P}_N \equiv A_1A_2 \dots A_N$, constructed by joining its vertices in even (or odd) positions,

5. Interior angles of a semi-regular polygon at odd vertices are marked with α , and those at even vertices are marked with β

6. Isosceles triangles

$$\Delta A_1A_2A_3, \Delta A_3A_4A_5, \dots, \Delta A_{2n-1}A_{2n}A_1$$

are triangles constructed over each side of an "inscribed" regular n -triangle.

7. r is the radius of the inscribed circle to the semi-regular equilateral polygon (Figure 1).

In addition to these interpreted marks, all other marks will

be interpreted when mentioned in a given definition.

To a semi-regular equilateral polygon $\mathcal{P}_N \equiv A_1 A_2 \dots A_N$ with $N = 2 \cdot n, n \geq 2, n \in \mathbb{N}$ and with equal sides there can be "inscribed" regular n -side polygons by joining odd vertices $\mathcal{P}_n^1 \equiv A_1 A_3 A_5 \dots A_{2n-3} A_{2n-1}$, or even vertices $\mathcal{P}_n^2 \equiv A_2 A_4 A_6 \dots A_{2n-2} A_{2n}$.

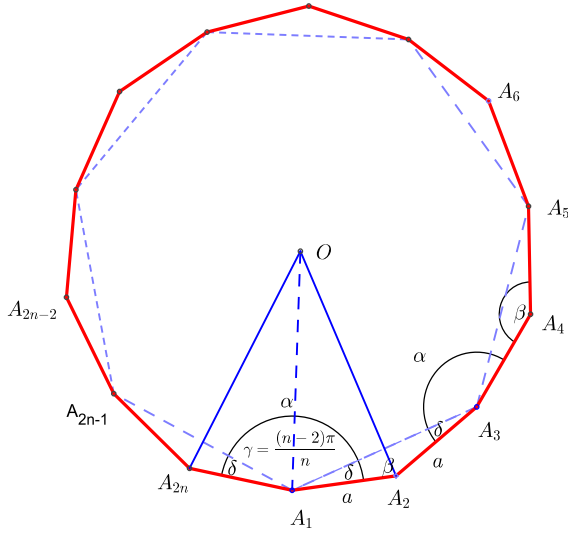


Figure 1. Basic elements of equilateral semi-regular polygon $\mathcal{P}_{2n}^{a, \delta}$ of a side a and angle δ .

To analyze the metric properties of regular polygons, it is sufficient for us to know one basic element, i.e. the length of a side, while for the semi-regular polygons this is not sufficient [2].

Therefore, in addition to side a of a semi-regular polygon, for the analysis of the metric properties we will use another element of it, and that is the angle between side a of the semi-regular polygon and side b of its "inscribed" regular polygon, which we mark with δ , i.e. $\delta = \angle(a, b)$ (Figure 1) [3].

To show that a semi-regular equilateral $2n$ -sides polygon is given by side a and angle δ , we write: $\mathcal{P}_{2n}^{a, \delta}$.

If $\gamma = \frac{(n-2)\pi}{n}$, $n \geq 2$ is the interior angle of the "inscribed" regular polygon \mathcal{P}_n^1 , then $\alpha = \gamma + 2\delta = \frac{(n-2)\pi}{n} + 2\delta$ gives interior angles at odd vertices, and $\beta = \pi - 2\delta$ gives the ones at even vertices of the semi-regular polygon \mathcal{P}_{2n}^2 of a side a , where $\delta = \angle(b, a)$ marks the angle between the sides of polygons \mathcal{P}_n^1 and \mathcal{P}_{2n} (Figure 1). Here, we consider that a regular polygon with $n = 2$ sides (segment) is "inscribed" to a semi-regular equilateral quadrilateral (rhombus).

Next, we consider those values of angle δ for which \mathcal{P}_N is a convex semi-regular equilateral polygon. We find the values of angle δ for which semi-regular equilateral polygon $\mathcal{P}_{2n}, n \geq 2, n \in \mathbb{N}$ is convex from the inequality which connects the definition of convexity with the values of interior angles of the semi-regular polygon [2].

That is, from inequality $\alpha < \pi$ i $\beta < \pi$ and values of angles α, β we find that for all $\delta \in \langle 0, \frac{\pi}{n} \rangle$ a semi-regular equilateral polygon \mathcal{P}_{2n} is convex, while for $\delta = \frac{2\pi}{n}$ it is

convex and regular. The following is true:

An equilateral semi-regular $2n$ -sides polygon of a side a and angle δ is:

1. convex, if $\delta \in \langle 0, \frac{\pi}{n} \rangle$. For $\delta = \frac{\pi}{2n}$ it is regular and convex.
2. non-convex, if $\delta \in \langle \frac{\pi}{n}, \frac{\pi}{2} \rangle$.
3. if $\delta > \frac{\pi}{2}$, then semi-regular polygon \mathcal{P}_{2n} is not defined [8-11].

2. My Result

2.1. Surface of Semi-Regular Equilateral Polygon $\mathcal{P}_{2n}^{a, \delta}$

Let us show that the surface area of $\mathcal{P}(a, \delta)$ equilateral semi-regular $2n$ -sides polygon as a function of a side a and angle δ is calculated by the formula (1).

Theorem 1. The surface area of equilateral semi-regular $2n$ -side polygon of side a and angle δ is calculated by the of formula

$$\mathcal{P}(a, \delta) = na^2 \frac{\cos \delta}{\sin \frac{\pi}{n}} \cos(\frac{\pi}{n} - \delta), \quad (1)$$

where n is the number of sides of the "inscribed" regular polygon $n \geq 2, n \in \mathbb{N}$.

Proof: Note that for the surface area of the inscribed equilateral semi-regular polygon $\mathcal{P}_{2n}^{a, \delta}$ the following equality is valid

$$\mathcal{P}(a, \delta) = \mathcal{A}(P_n^b) + n\mathcal{A}(P_3) \quad (2)$$

where $\mathcal{A}(P_n^b)$ is the surface area of the inscribed regular n -sides polygon of a side b , and $\mathcal{A}(P_3)$ is the surface area of the isosceles triangle.

Let us further note a fragment of a semi-regular polygon (Figure 2). It follows from a special right triangle $\triangle A_1 A_2 K$ that $b = 2a \cos \delta$, so the surface area of an isosceles triangle is $\mathcal{A}(P_3) = a^2 \sin \delta \cos \delta$.

Further, for the surface area of the "inscribed" polygon \mathcal{P}_n , according to the markings in Figure 2, the following is valid

$$\mathcal{A}(P_n^b) = \frac{n}{4} b^2 \cot \frac{\alpha}{2} = \frac{n}{4} (2a \cos \delta)^2 \cot \frac{\pi}{n} = na^2 \cos^2 \delta \cot \frac{\pi}{n}.$$

Based on that, the surface area of a semi-regular equilateral polygon \mathcal{P}_{2n} is

$$\begin{aligned} \mathcal{P}(a, \delta) &= na^2 \cos^2 \delta \cot \frac{\pi}{n} + na^2 \sin \delta \cos \delta \\ &= na^2 \cos \delta (\cos \delta \cot \frac{\pi}{n} + \sin \delta) = \dots = na^2 \frac{\cos \delta}{\sin \frac{\pi}{n}} \cos(\frac{\pi}{n} - \delta). \end{aligned}$$

Corollary 2. Notice that for $\delta = \frac{\pi}{2n}$ $2n$ -sides polygon becomes a regular polygon and the formula for calculating the surface area is

$$\mathcal{P}(a, \delta) = na^2 \frac{\cos \frac{\pi}{2n}}{\sin \frac{\pi}{n}} \cos(\frac{\pi}{n} - \frac{\pi}{2n})$$

$$\begin{aligned}
 &= na^2 \frac{\cos \frac{\pi}{2n}}{2 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}} \cos \frac{\pi}{2n} \\
 &= \frac{na^2}{2} \cot \frac{\pi}{2n} = \frac{(2n)a^2}{4} \cot \frac{\pi}{2n}.
 \end{aligned}$$

The latter equality is a formula for calculating the surface area of a regular $2n$ -sides polygon.

If in the formula for calculating the surface area of a semi-regular polygon we include that $n = 2$, then we get that

$$\begin{aligned}
 \mathcal{P}(a, \delta) &= 2 \cdot a^2 \cdot \frac{\cos \delta}{\sin \frac{\pi}{2}} \cdot \cos \left(\frac{\pi}{2} - \delta \right) \\
 &= 2a^2 \cos \delta \cos \left(\frac{\pi}{2} - \delta \right) \\
 &= a^2 \sin(2\delta),
 \end{aligned}$$

i.e. we get a formula for calculating the surface area of an equilateral quadrilateral (rhombus), where δ is the angle between the diagonal and the side of the rhombus.

2.2. Radius of the Inscribed Circle

For a semi-regular equilateral convex $2n$ -sides polygon of a side a and angle δ to which the circle of radius r can be inscribed, the following theorem holds:

Theorem 3. *The radius of a circle inscribed to a semi-regular equilateral polygon $\mathcal{P}_{2n}^{a,\delta}$ depending on a side a and angle δ is calculated by the following formula*

$$r = a \frac{\cos \delta \cos \left(\frac{\pi}{n} - \delta \right)}{\sin \frac{\pi}{n}}, \quad (3)$$

where n is the number of sides of the "inscribed" regular polygon and $n \geq 2$ and $n \in \mathbb{N}$.

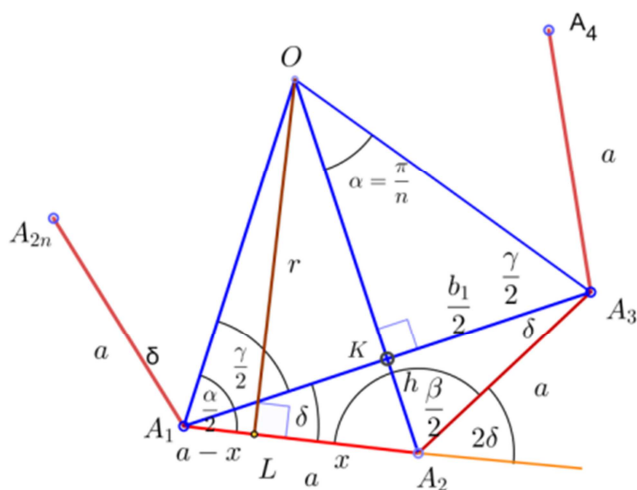


Figure 2. A fragment of $2n$ -sides polygon.

Proof: From a special right triangle $\triangle OA_1L$ (Figure 2) we have that $\tan \frac{\alpha}{2} = \frac{r}{a-x}$ and from it $x = a - \frac{r}{\tan \frac{\alpha}{2}}$. Similarly, from a special right triangle $\triangle OA_2L$ it follows that $\tan \frac{\beta}{2} = \frac{r}{x}$.

If we include the calculated value for x in the latter equality we find that

$$r = \left(a - \frac{r}{\tan \frac{\alpha}{2}} \right) \tan \left(\frac{\pi}{2} - \delta \right).$$

If we express radius r from this equality, after calculation, we find that

$$r = \frac{a \cdot \tan \frac{\beta}{2}}{1 + \frac{\tan \frac{\beta}{2}}{\tan \frac{\alpha}{2}}}.$$

Since,

$$\begin{aligned}
 \frac{\alpha}{2} &= \frac{(n-2)\pi}{2n} + \delta \text{ and } \frac{\beta}{2} \\
 &= \frac{\pi}{2} - \delta \text{ and } \tan \left(\frac{\pi}{2} - \frac{\pi}{n} \right) = \cot \frac{\pi}{n},
 \end{aligned}$$

from the latter equality, after replacement, we find that the radius of the inscribed circle can be obtained by the formula

$$r = a \frac{\cot \delta \cot \left(\frac{\pi}{n} - \delta \right)}{\cot \delta + \cot \left(\frac{\pi}{n} - \delta \right)}. \quad (4)$$

From this equality, we obtain the required form

$$r = a \frac{\cos \delta \cos \left(\frac{\pi}{n} - \delta \right)}{\sin \frac{\pi}{n}}.$$

Corollary 4. *If in the latter equality we put that $\delta = \frac{\pi}{n}$, we get that $r = a \cot \frac{\pi}{n}$, i.e. the radius of the inscribed circle of a regular \mathcal{P}_{2n} polygon.*

Corollary 5. *If we write equality*

$$r = a \frac{\cos \delta \cos \left(\frac{\pi}{n} - \delta \right)}{\sin \frac{\pi}{n}} \text{ in a form of } \frac{r}{a} = \frac{\cos \delta \cos \left(\frac{\pi}{n} - \delta \right)}{\sin \frac{\pi}{n}}$$

and substitute it in the formula for the surface area of a semi-regular equilateral $2n$ -sides polygon, after calculation we get the following:

$$\mathcal{P}(a, \delta) = na^2 \frac{r}{a} = n \cdot a \cdot r. \quad (5)$$

From this it follows that the surface area of a semi-regular equilateral polygon \mathcal{P}_{2n} is equal to the product of the length of the side, the radius of the inscribed circle, and the number of sides of the inscribed regular n -sides polygon.

Also, if in equality $r = a \frac{\cos \delta \cos \left(\frac{\pi}{n} - \delta \right)}{\sin \frac{\pi}{n}}$ we put that $a = \frac{b}{2 \cos \delta}$, after calculation we find that the radius of the inscribed circle to a semi-regular equilateral polygon depending on a side b of a regular polygon \mathcal{P}_n^1 inscribed to it and angle $\delta = \angle(a, b)$, can be expressed by the following relation

$$r = \frac{b}{2} \frac{\cos \left(\frac{\pi}{n} - \delta \right)}{\sin \frac{\pi}{n}}. \quad (6)$$

Proposition 5. *The surface area of a semi-regular equilateral polygon with $N = 2n, n \geq 2, n \in \mathbb{N}$, sides can be calculated by the following formula*

$$\mathcal{P}_{2n} = \frac{nbr}{2\cos\delta} \quad (7)$$

where b is the side of the inscribed regular n -sides polygon, r is the radius of the inscribed circle and $\delta = \angle(a, b)$ is the angle between the side of the semi-regular polygon and the inscribed semi-regular n -sides polygon.

Proof. Since $\frac{2r}{b} = \frac{\cos(\frac{\pi}{n}-\delta)}{\sin\frac{\pi}{n}}$ from the formula for the surface area of a semi-regular $2n$ -sides polygon

$$\mathcal{P}_{2n} = na^2 \frac{\cos\delta}{\sin\frac{\pi}{n}} \cos\left(\frac{\pi}{n} - \delta\right)$$

and based on the previous relations we have that

$$\begin{aligned} \mathcal{P}_{2n} &= na^2 \frac{\cos\delta}{\sin\frac{\pi}{n}} \cos\left(\frac{\pi}{n} - \delta\right) = na^2 \cos\delta \frac{2r}{b} \\ &= n \cos\delta \frac{2r}{b} \frac{b^2}{4 \cos^2\delta} = \frac{nrb}{2 \cos\delta}. \end{aligned} \quad (8)$$

2.3. A Constructive Task

Let us formulate and prove the following theorem, which is related to the problem of a semi-regular $2n$ -sides polygon and regular polygons inscribed therein.

Theorem 6. Let $\mathcal{A}_{2n} \equiv A_1 A_2 \dots A_{2n-1} A_{2n}$, $n > 2$ be a convex equilateral semi-regular $2n$ -sides polygon and $\mathcal{P}_n^1 \equiv A_1 A_3 A_5 \dots A_{2n-3} A_{2n-1}$, $\mathcal{P}_n^2 \equiv A_2 A_4 A_6 \dots A_{2n-2} A_{2n}$ be regular n -sides polygons inscribed therein, formed by joining the vertices at odd or even positions. *To be proven:*

1. the intersection points of the sides of these regular polygons are the vertices of an equiangular semi-regular $2n$ -sides polygon $\mathcal{B}_{2n} \equiv B_1 B_2 B_3 \dots B_{2n-1} B_{2n}$,
2. the lengths of the sides of that equiangular polygon \mathcal{B}_{2n} relate as the sine of the angles closed by the sides of regular polygons with the side of a semi-regular equilateral $2n$ -sides polygon \mathcal{A}_{2n} .

Proof. *a)* Let there be a fragment of a semi-regular equilateral $2n$ -sides polygon

$$\mathcal{A}_{2n} \equiv A_1 A_2 \dots A_{2n} \text{ of a side } a \text{ and}$$

$$\mathcal{P}_n^1 \equiv A_1 A_3 A_5 \dots A_{2n-3} A_{2n-1}$$

$$\mathcal{P}_n^2 \equiv A_2 A_4 A_6 \dots A_{2n-2} A_{2n}$$

and let here be a regular n -sides polygon of sides b_1 , b_2 respectively, inscribed to it, and let $\delta = \angle(a, b_1)$ be the angle between a side a and a side b_1 of a regular n -sides polygon \mathcal{P}_n^1 . Let $B_1 B_2 = c_1, B_2 B_3 = c_2, \dots, B_{2n-1} B_{2n} = c_1, B_{2n} B_1 = c_2$ be sides of a $2n$ -sides polygon $\mathcal{B}_{2n} \equiv B_1 B_2 B_3 \dots B_{2n-1} B_{2n}$ constructed by the intersection of the sides of regular polygons \mathcal{P}_n^1 and \mathcal{P}_n^2 (Figure 3).

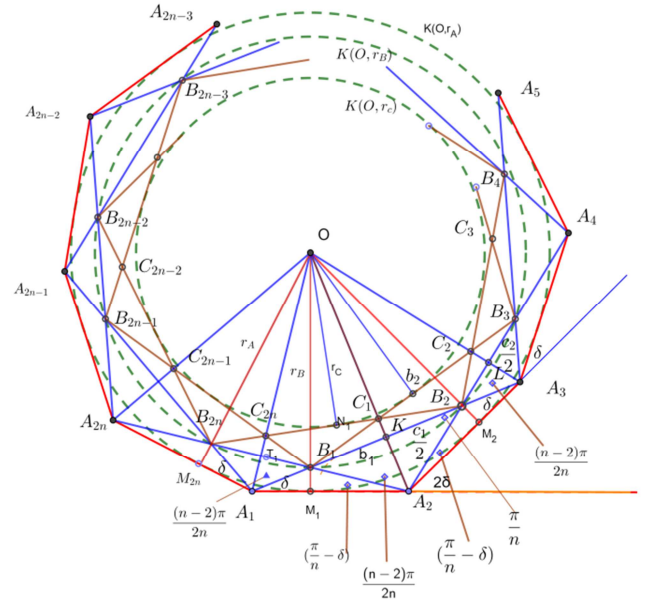


Figure 3. Fragment of a semi-regular $2n$ -sides polygon with basic elements.

Let us first prove that a constructed $2n$ -sides polygon $\mathcal{B}_{2n} \equiv B_1 B_2 B_3 \dots B_{2n-1} B_{2n}$ has equal all interior angles. Note that the following is valid for the interior angles of a semi-regular equilateral polygon \mathcal{A}_{2n} at vertices in odd positions,

$$\begin{aligned} \angle A_1 &= \angle A_3 = \dots \\ &= \angle A_{2n-3} = \angle A_{2n-1} \\ &= \alpha = \frac{(n-2)\pi}{n} + 2\delta \end{aligned} \quad (9)$$

and for the interior angles at vertices in even positions, the following is valid

$$\begin{aligned} \angle A_2 &= \angle A_4 = \dots \\ &= \angle A_{2n-2} = \angle A_{2n} \\ &= \beta = \frac{(n-2)\pi}{n} + 2\left(\frac{\pi}{n} - \delta\right) = \pi - 2\delta \end{aligned} \quad (10)$$

Since

$$A_1 A_2 = A_2 A_3 = \dots = a_{2n-1} A_{2n} = A_{2n} A_1 = a$$

and $\angle A_1 = \angle A_3 = \dots = \angle A_{2n-3} = \angle A_{2n-1} = \delta$ and $\angle A_2 = \angle A_4 = \dots = \angle A_{2n-2} = \angle A_{2n} = \frac{\pi}{n} - \delta$, triangles $\triangle A_1 B_1 A_2, \triangle A_2 B_2 A_3, \dots, \triangle A_{2n} B_{2n} A_1$ are congruent, and, hence their interior angles are also congruent, i.e. the following is valid:

$$\angle A_1 B_1 A_2 \cong \angle A_2 B_2 A_3 \cong \dots$$

$$\cong \angle A_{2n} B_{2n} A_1 = \frac{(n-1)\pi}{n} \quad (11)$$

Note that the interior angles of a $2n$ -sides polygon $\mathcal{B}_{2n} \equiv B_1 B_2 B_3 \dots B_{2n-1} B_{2n}$ are cross-sectional to angles in (11), therefore, they are congruent with them, and it follows that

$$\angle B_i = \frac{(n-1)\pi}{n}, i = 1, 2, \dots, 2n, n \in \mathbb{N} \quad (12)$$

It follows from (12) that a $2n$ -sides polygon $\mathcal{B}_{2n} \equiv B_1B_2B_3 \dots B_{2n-1}B_{2n}$ is equiangular. Let us prove that an equiangular $2n$ -sides polygon

$\mathcal{B}_{2n} \equiv B_1B_2B_3 \dots B_{2n-1}B_{2n}$ has different sides. Note that triangles $\Delta B_1A_2B_2, \Delta B_3A_4B_4, \dots, \Delta B_{2n-1}A_{2n}B_{2n}, \Delta B_{2n-1}A_1B_1$ are congruent with each other because of the following:

$\angle B_1 = \angle B_2 = \dots = \angle B_{2n-1} = \angle B_{2n} = \frac{\pi}{n}$ and $\angle A_2 = \angle A_4 = \dots = \angle A_{2n-2} = \angle A_{2n} = \frac{(n-2)\pi}{n}$, and from the congruency of triangles

$$\Delta A_1B_1A_2, \Delta A_2B_2A_3, \dots, \Delta A_{2n}B_{2n}A_1$$

the congruency of the sides follows. Similarly, we conclude that triangles

$\Delta B_2A_3B_3, \Delta B_4A_5B_5, \dots, \Delta B_{2n-2}A_{2n-1}B_{2n-1}, \Delta B_{2n}A_1B_1$ are also mutually congruent because $\angle B_2 = \angle B_3 = \dots = \angle B_{2n-1} = \angle B_{2n} = \frac{\pi}{n}$ and $\angle A_1 = \angle A_3 = \dots = \angle A_{2n-1} = \frac{(n-2)\pi}{n}$, and from the congruency of triangles $\Delta A_1B_1A_2, \Delta A_2B_2A_3, \dots, \Delta A_{2n}B_{2n}A_1$ follows the equality of sides, and also the diversity of their bases

$$B_1B_2 = c_1, B_2B_3 = c_2, \dots, B_{2n-1}B_{2n} = c_1,$$

$B_{2n}B_1 = c_2$ because these triangles are not isosceles. In doing so, we have proved that a $2n$ -sides polygon $\mathcal{B}_{2n} \equiv B_1B_2B_3 \dots B_{2n-1}B_{2n}$ is an equiangular semi-regular polygon.

b) In general, observe a triangle $\Delta A_1A_2A_3$ with the given marks (Figure 3). For the interior angles of a triangle it is true that $\angle A_1 = \angle A_3 = \delta$, $\angle A_2 = \frac{(n-2)\pi}{n} + 2\left(\frac{\pi}{n} - \delta\right)$ and that $A_1A_2B_1 = \frac{\pi}{n} - \delta$. From the special right angle triangle ΔA_1KA_2 we find that the side of the inscribed n -sides polygon is \mathcal{P}_n^1 , $b_1 = 2a \cos \delta$, and from the special right angle triangle ΔA_2KB_1 we find that $\angle A_2 = \frac{(n-2)\pi}{n}$ and $\angle B = \frac{\pi}{n}$, as well as that one side of a $2n$ -sides polygon - \mathcal{B}_{2n} is given in a relation

$$c_1 = \frac{2a \sin \delta}{\tan \frac{\pi}{n}}, n > 2, n \in \mathbb{N} \quad (13)$$

because for $n = 2$ we have a rhombus to which it is not possible to inscribe a semi-regular equiangular polygon.

From equiangular triangle ΔB_2LA_3 we find that the side of the inscribed regular \mathcal{P}_n^2 is given in a relation

$$b_2 = 2a \cdot \cos\left(\frac{\pi}{n} - \delta\right) \quad (14)$$

and that the other side of equiangular $2n$ -sides polygon \mathcal{B}_{2n} is given in a relation

$$c_2 = \frac{2a \sin\left(\frac{\pi}{n} - \delta\right)}{\tan \frac{\pi}{n}}, n > 2, n \in \mathbb{N} \quad (15)$$

Based on relations (13) and (15), we find that

$$\frac{c_2}{c_1} = \frac{\frac{2a \sin\left(\frac{\pi}{n} - \delta\right)}{\tan \frac{\pi}{n}}}{\frac{2a \sin \delta}{\tan \frac{\pi}{n}}} = \frac{\sin\left(\frac{\pi}{n} - \delta\right)}{\sin \delta} \blacksquare. \quad (16)$$

On the right-hand side of this formula (16), note the function $f(n, \delta) = \frac{\sin\left(\frac{\pi}{n} - \delta\right)}{\sin \delta}$ and select the value of angle δ such that it is possible to construct it geometrically, e.g. in the form of $\delta = \frac{\pi}{2n}$.

Note that from the Bernoulli's inequality $(1+x)^n \geq 1+nx$, (which for $x = 1$, transforms into a form of $2^n \geq 1+n$), it follows that inequalities $\frac{\pi}{2^n} \leq \frac{\pi}{n+1} < \frac{\pi}{n}$ are valid for every natural number $n \in \mathbb{N}$, and thus inequalities

$$\frac{\pi}{2^n} = \frac{\pi}{2 \cdot 2^{n-1}} = \frac{\frac{\pi}{2^{n-1}}}{2} < \frac{\pi}{2^{n-1}} \leq \frac{\pi}{n}$$

are also valid for $n \in \mathbb{N}$. Therefore, let the value of angle δ be given by relation $\delta(n) = \frac{\pi}{2^{n-1}} \in \left(0, \frac{\pi}{n}\right)$ for $n > 2, n \in \mathbb{N}$ then function $f(n, \delta)$ takes the form of:

$$f(n) = \frac{\sin\left(\frac{\pi}{n} - \frac{\pi}{2^{n-1}}\right)}{\sin \frac{\pi}{2^{n-1}}}, n \in \mathbb{N}. \quad (17)$$

Note that for $n = 1$ a function $f(1)$ is not defined, and for $n=2$, it is $f(2) = \frac{\sin(0)}{\sin \frac{\pi}{2}} = 0$, which is of no interest here given the nature of the function.

Example: For $n = 3$ it is $\delta(3) = \frac{\pi}{4}$ and

$$f(3) = \frac{\sin\left(\frac{\pi}{3} - \frac{\pi}{2^2}\right)}{\sin \frac{\pi}{2^2}} = \frac{\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}{\sin \frac{\pi}{4}} = \frac{\sin \frac{\pi}{12}}{\sin \frac{\pi}{4}} = \sqrt{2} \sin \frac{\pi}{12} = \sqrt{2} \cdot \frac{1}{2} \sqrt{2 - \sqrt{3}} = \frac{1}{2} \sqrt{4 - 2\sqrt{3}}.$$

From relation (16), it follows that the ratio of the sides of a semi-regular equiangular hexagon inscribed to a semi-regular equilateral hexagon is $\frac{c_2}{c_1} = \frac{1}{2} \sqrt{4 - 2\sqrt{3}}$. For $n = 4$ it is $\delta(4) = \frac{\pi}{8}$, and the value of the function is

$$f(4) = \frac{\sin\left(\frac{\pi}{4} - \frac{\pi}{2^{4-1}}\right)}{\sin \frac{\pi}{2^{4-1}}} = \frac{\sin \frac{\pi}{8}}{\sin \frac{\pi}{8}} = 1,$$

so $\frac{c_2}{c_1} = 1 \Rightarrow c_2 = c_1$. So it follows that $\mathcal{B}_{2n}^{c_1, c_2}$ and $\mathcal{A}_{2n}^{a, \delta}$ are regular polygons with internal angles of $\frac{3\pi}{4}$, that is, the following proposition is valid:

Proposition 7. For the value of angle $\delta(n)$ expressed in a formula: $\delta(n) = \frac{\pi}{2^{n-1}}, n > 2, n \in \mathbb{N}$ semi-regular equiangular polygon $\mathcal{B}_{2n}^{c_1, c_2}$ is regular, if and only if $n = 4$.

Proof: Let us suppose that equiangular $2n$ -sides polygon $\mathcal{B}_{2n}^{c_1, c_2}$ is a regular polygon. Then it is also true that $c_1 = c_2$. Further from the requirement $f(n) = 1$ we have the

following equality $\frac{\sin\left(\frac{\pi}{n} - \frac{\pi}{2^{n-1}}\right)}{\sin \frac{\pi}{2^{n-1}}} = 1$ which, after calculation, becomes: $2 \cos \frac{\pi}{2^n} \cdot \sin\left(\frac{\pi}{n} - \frac{\pi}{2^{n-1}}\right) = 0 \Leftrightarrow \sin\left(\frac{\pi}{n} - \frac{\pi}{2^{n-1}}\right) = 0 \Leftrightarrow 2^{n-1} = 2n$.

In the set of natural numbers, only solution $n = 4$ meets the requirement $n > 2$.

3. Conclusion

Based on the content presented in this paper, it follows that

the basic metric properties, such as: the convexity, the surface area of a semi-regular $2n$ -sides polygon, the radius of the inscribed circle and the ratios of a semi-regular $2n$ -sides polygon can be expressed by side a and angle δ .

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