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Boundedness of Littlewood-Paley Operators in Variable Morrey Spaces

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Abstract: In this paper, the authors prove norm inequalities for the intrinsic square functions and commutators generated by this class operator and BMO function in variable Morrey spaces. This implies that the same norm inequalities for the Lusin area integrals, the Littlewood-Paley operators and the continuous square functions. As application, we get the boundedness for convolution Calderón-Zygmund operators in generalized Morrey spaces.

Keywords: Littlewood-Paley Operators, Singular Integrals, Morrey Spaces, Commutator

1. Introduction

There is a greatly interested to study variable exponent spaces and operators with variable parameters in last two decade. Some researches are monographs about Lebesgue spaces with variable exponents, for example, [1, 2]. Let $p(\cdot)$ be a measurable function : $\Omega \to [1, \infty)$. We suppose that

$$1 < p_- \leqslant p(\cdot) \leqslant p_+ < \infty, \tag{1}$$

where $p_- = \operatorname{ess\,inf}_{x\in\Omega} p(x)$, $p_+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$.

We let $\mathscr{L}^{p(\cdot)}(\Omega)$ be the set of functions f such that

$$\rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}dx < \infty.$$

It is a Banach space equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 : \rho_{p(\cdot)}(f/\eta) \leqslant 1\right\},$$

we denote the conjugate exponent by $p'(x) = \frac{p(x)}{p(x)-1}$ for $x \in \Omega$.

The Hölder inequality is valid in the form

$$\int_\Omega |f(x)g(x)|dx \leqslant \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

Definition 1.1. Let Ω is open bounded set. We say that $p(\cdot)$ satisfies the log-Hölder condition, and denote this by $p(\cdot) \in LH(\Omega)$, if there exists a constant C such that for all $x, y \in \Omega$, $|x-y| < 1/2$,

$$|p(x) - p(y)| \leqslant \frac{C}{-\ln|x-y|}.$$

Let M be the Hardy-Littlewood maximal operator, i.e., for $f \in L^1_{loc}(\mathbb{R}^n)$

$$Mf(x) = \sup \frac{1}{|Q|}\int_Q |f(y)|dy,$$

where the supremum is taken over all cubes Q containing x, and the $|Q|$ is the Lebesgue measure of $Q \subset \mathbb{R}^n$. Let $\mathscr{B}(\Omega)$ be the set such that M is bounded on $\mathscr{L}^{p(\cdot)}(\Omega)$. Diening [3] proved that $p(\cdot) \in \mathscr{B}(\Omega)$ if $p(\cdot) \in LH(\Omega)$, i.e M is bounded on $\mathscr{L}^{p(\cdot)}(\Omega)$. When Ω is unbounded to see [4].

Morrey spaces play an important role in study of local properties of solutions of partial differential equations. These spaces were introduced by Morrey [11] in 1938. The norm of Morrey space is defined as follows:

$$\|f\|_{p,\lambda} = \sup_{x,\,r>0}\left(\frac{1}{r^\lambda}\int_Q |f(y)|^p dy\right)^{1/p},$$

where Q is the cube with the center x and with side-length r and its sides parallel to the coordinate axes, $0 < \lambda < n$ and $1 \leqslant p < \infty$. Variable exponent Morrey spaces were firstly introduced in [13] and also studied in [12]. Generalized variable exponent Morrey spaces $\mathscr{L}^{p(\cdot),\omega}(\Omega)$ were introduced in [6]. They proved the boundedness of maximal, singular, potential operator in variable exponent Morrey spaces $\mathscr{L}^{p(\cdot),\omega}(\Omega)$. When $\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$, the $\mathscr{L}^{p(\cdot),\omega}(\Omega)$ are the variable exponent Morrey spaces $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ in [8].

At the same time, the authors of [6] introduced another generalized variable exponent Morrey spaces $\mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ in [7].

In this paper, we denotes $\widetilde{Q} = Q \cap \Omega$. Let $\omega(x,r)$ be a positive measurable function on $\Omega \times (0,\ell)$, $\ell = \operatorname{diam}\Omega$ and $1 \leqslant p < \infty$.

The generalized variable exponent Morrey space $\mathscr{L}^{p(\cdot),\omega}(\Omega)$ is equipped with the norm

$$\|f\|_{p(\cdot),\omega} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))},$$

where we assume that

$$\inf_{x\in\Omega, r>0} \omega(x,r) > 0. \tag{2}$$

It is the variable Morrey space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$, where $\lambda(x)$ is a measurable function on Ω with values in $[0,n]$. When p is constant, the norm is defined by

$$\|f\|_{p,\omega} = \sup_{x\in\Omega, r>0} \frac{1}{\omega(x,r)^{1/p}} \|f\|_{L^p(\widetilde{Q}(x,r))},$$

In this paper we mainly consider the boundedness of intrinsic square functions in variable Morrey spaces. Let $u(x,t) = P_t * f(x)$ be the Poisson integral of f, where $P_t(x) = c_n \frac{t}{(t^2+|x|^2)^{(n+1)/2}}$ with $c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$.

The cone of aperture β for any $\beta > 0$ is defined by

$$\Gamma_\beta(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |y-x| < \beta t\}.$$

The corresponding Lusin area integral S_β is defined by

$$S_\beta(f)(x) = \left(\int_{\Gamma_\beta(x)} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2}.$$

The Littlewood-Paley g-function and g^*_λ-function are defined respectively by

$$g(f)(x) = \left(\int_0^\infty t|\nabla u(x,t)|^2 dt \right)^{1/2}$$

and

$$g^*_\lambda(f)(x) = \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2}, \quad \lambda > 1.$$

Let $\psi_t(x) = t^{-n}\psi(x/t)$ and $\psi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ be real, radial, supported in $\{x : |x| \leq 1\}$ and

$$\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} = 1$$

for all $\xi \neq 0$, where $\widehat{\psi}$ denotes the Fourier transform of ψ.

The continuous square functions $S_{\psi,\beta}$ and Littlewood-Paley g-function g_ψ are defined by

$$S_{\psi,\beta}(f)(x) = \left(\int_{\Gamma_\beta(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

and

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(y)|^2 \frac{dt}{t} \right)^{1/2}.$$

For $0 < \alpha \leq 1$, let $\mathcal{C}_\alpha$ be the family of function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that it is supported in $\{x : |x| \leq 1\}$, $\int \phi(x)dx = 0$, and for all x and x'

$$|\phi(x) - \phi(x')| \leq |x-x'|^\alpha.$$

For $f \in L^1_{loc}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^{n+1}_+$, set

$$A_\alpha(f)(y,t) = \sup_{\phi\in\mathcal{C}_\alpha} |f * \phi_t(y)|.$$

In [16], the intrinsic square function is defined by

$$G_{\beta,\alpha}(f)(x) = \left(\int_{\Gamma_\beta(x)} (A_\alpha(f)(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

when $\beta = 1$, denote $G_{1,\alpha}$ by G_α.

The intrinsic Littlewood-Paley g-function and the intrinsic g^*_λ-function are defined respectively by

$$g_\alpha(f)(x) = \left(\int_0^\infty (A_\alpha(f)(x,t))^2 \frac{dt}{t} \right)^{1/2}$$

and

$$g^*_{\lambda,\alpha}(f)(x) = \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} (A_\alpha(f)(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad \lambda > 1.$$

We recall several properties of the intrinsic square function, the proofs of them to see [16, 17].

(1) G_α is of weak type (1,1):

$$|\{x \in \mathbb{R}^n : G_\alpha(f)(x) > \sigma\}| \leqslant \frac{C(n,\alpha)}{\sigma} \|f\|_{L^1(\mathbb{R}^n)}; \tag{3}$$

(2) If $\beta \geqslant 1$, then for all $x \in \mathbb{R}^n$,

$$G_{\beta,\alpha}(f)(x) \leqslant C(\alpha,\beta,n) G_\alpha(f)(x); \tag{4}$$

(3) If S is anyone the Littlewood-Paley operators defined above, then

$$S(f)(x) \leqslant C G_\alpha(f)(x), \tag{5}$$

where the constant C is independent of f and x;

(4) The function G_α and g_α are pointwise comparable with comparability constants only depending on α and n.

A locally integrable function b belongs to BMO if

$$\|b\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx < \infty,$$

where $b_Q = \frac{1}{|Q|} \int_Q b(x) dx$ and the supremum is taken over all cube Q whose sides parallel to the coordinate axes in $\mathbb{R}^n$.

The commutators generated by BMO function b and intrinsic square functions are respectively defined by

$$[b, G_\alpha](f)(x) = \left(\int_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y-z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

$$[b, g_\alpha](f)(x) = \left(\int_0^\infty \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(x-z) f(z) dz \right|^2 \frac{dt}{t} \right)^{1/2}$$

and

$$[b, g^*_{\lambda,\alpha}](f)(x) = \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y-z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.$$

In order to study the commutators, we need the following properties of BMO. For $b \in \mathrm{BMO}$ and $1 < p < \infty$, we get

$$\|b\|_{\mathrm{BMO}} \sim \sup_Q \left(\frac{1}{|Q|} \int\limits_Q |b(x) - b_Q|^p dx \right)^{1/p} \tag{6}$$

For all nonnegative integers l, we obtain

$$|b_{2^{l+1}Q} - b_Q| \leq C(l+1) \|b\|_{\mathrm{BMO}}. \tag{7}$$

Wang [19] has estimated this class operator for ω is increasing and there is a constant D, $1 \leqslant D < 2^n$, such that, for any $r > 0$, $\omega(2Q) \leqslant D\omega(Q)$, where $\omega(Q) = \omega(x,r)$, Q is the cube with the center x and with side-length r and its sides parallel to the coordinate axes. We can impose the following condition on $\omega(x,r)$ as in [10].

Assume that there is a constant C such that, for any $x \in \mathbb{R}^n$ and for any $r > 0$,

$$1/C \leqslant \omega(x,t)/\omega(x,r) \leqslant C, \quad where\ r \leqslant t \leq 2r \tag{8}$$

and

$$\int_r^{\infty} \frac{\omega(x,t)}{t^{n+1}} dt \leqslant C \frac{\omega(x,r)}{r^n}. \tag{9}$$

Theorem 1.1. Assume that $\omega(x,r)$ satisfy the conditions (8) and (9). Then *(i)* there is a constant $C_p > 0$ such that

$$\|S(f)\|_{p,\omega} \leqslant C_p \|f\|_{p,\omega} \tag{10}$$

for $f \in L^{p,\omega}$ and $1 < p < \infty$;

(ii) there is a constant $C_p > 0$ such that for any $\sigma > 0$ and any cube Q,

$$\frac{|\{x \in Q : S(f)(x) > \sigma\}|}{\omega(Q)} \leqslant \frac{C_p}{\sigma^p} \|f\|_{p,\omega}^p \tag{11}$$

for $f \in L^{p,\omega}$ and $1 < p < \infty$.

Based on Theorem 1.1, (10) holds for the classical Calderón-Zygmund operators along with the Littlewood-Paley technique in [15, 16, 17] developed by Wilson. Let

$$Tf(x) = \lim_{\varepsilon \to 0} T_\varepsilon f(x) = \lim_{\varepsilon \to 0} f * K_\varepsilon(x) \text{ and } T^* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|,$$

Where $K_\varepsilon(x) = K(x)\chi_{\{|x|>\varepsilon\}}$ and K is the standard kernel with following properties

1. $|K(x)| \leqslant \frac{C}{|x|^n}$;
2. $\int_{r<|x|<R} K(x)dx = 0,\ 0 < r < R < \infty$;
3. $|K(x) - K(x-y)| \leqslant \frac{C|y|^\delta}{|x|^{n+\delta}},\ |x| \geqslant 2|y|,\ \delta > 0.$

Corollary 1.2 If ω satisfies the conditions (8) and (9). Then for $1 < p < \infty$, there is a constant $C_p > 0$ such that (10) holds for T.

Corollary was also proved in [10].

Theorem 1.2. If ω satisfies the conditions (8) and (9). Then *(i)* there is a constant $C_p > 0$ such that (10) holds for $g^*_{\lambda,\alpha}$ where $0 < \alpha \leqslant 1$, $\lambda > 3$ and $1 < p < \infty$;
(ii) there is a constant $C_p > 0$ such that any cube Q, (11) holds for $g^*_{\lambda,\alpha}$, where $0 < \alpha \leqslant 1$, $\lambda > \frac{3n+2\alpha}{n}$ and $1 \leq p < \infty$.

Theorem 1.3. If ω satisfies the conditions (8) and (9). Then for $0 < \alpha \leqslant 1$ and $1 < p < \infty$, there is a constant $C_p > 0$ such that (10) holds for $[b, G_\alpha]$.

Theorem 1.4. If ω satisfies the conditions (8) and (9). Then for $0 < \alpha \leqslant 1$, $\lambda > 3$ and $1 < p < \infty$, there is a constant $C_p > 0$ such that (10) holds for $[b, g^*_{\lambda,\alpha}]$.

We can obtain the following results by property (4), Theorem 1.1 and Theorem 1.3.

Corollary 1.6 If ω satisfies the conditions (8) and (9). Then *(i)* for $0 < \alpha \leqslant 1$ and $1 < p < \infty$, there is a constant $C_p > 0$ such that (10) holds for g_α;

(ii) for $0 < \alpha \leqslant 1$ and $1 \leqslant p < \infty$, there is a constant $C_p > 0$ such that for any $\lambda > 0$ and any cube Q (11) holds for g_α.

Corollary 1.7 If ω satisfies the conditions (8) and (9). Then for $0 < \alpha \leqslant 1$ and $1 < p < \infty$, there is a constant $C_p > 0$ such that (10) holds for $[b, g_\alpha]$.

We have the following conclusion on variable exponent generalized Morrey spaces $\mathscr{L}^{p(\cdot),\omega}(\Omega)$.

Theorem 1.5. Let Ω be an open bounded set, namely, $\ell = \operatorname{diam} \Omega < \infty$ and $p(\cdot) \in LH(\Omega)$ satisfy assumption (1) and the function $\omega_1(x,r)$ and $\omega_2(x,r)$ satisfy the condition

$$\int_r^{\ell} \omega_1(x,t) \frac{dt}{t} \leqslant C\omega_2(x,r),$$

where C in independent of x and t. Then G_α is bounded from $\mathscr{L}^{p(\cdot),\omega_1}(\Omega)$ to $\mathscr{L}^{p(\cdot),\omega_2}(\Omega)$.

Corollary 1.9 Let Ω be an open bounded set and $p(\cdot) \in LH(\Omega)$ satisfy assumption (1), $\lambda(x) \geqslant 0$ and $\sup_{x\in\Omega} \lambda(x) < n$. Then G_α is bounded in the space $\mathscr{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

In the following, we introduce another generalized the variable exponent Morrey spaces $\mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$.

Definition 1.2 Let $\omega(x,r) : \Omega \times (0,\ell) \to \mathbb{R}^+$ and $\theta(r) : (0.\ell) \to [1,\infty]$ be measurable functions. The space $\mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ is set of functions with the finite norm

$$\|f\|_{\mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)} = \sup_{x\in\Omega} \left\| \frac{\omega(x,r)}{r^{\frac{n}{p(x)}}} \|f\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))} \right\|_{L^{\theta(\cdot)}(0,\ell)}.$$

The generalized Morrey space $\mathscr{L}^{p,\theta,\omega}$ with constant exponent first appeared in [11]. If $\theta(r) = \infty$, we can write

$$\mathscr{L}^{p(\cdot),\infty,\omega(\cdot)}(\Omega) = \mathscr{L}^{p(\cdot),\frac{1}{\omega(\cdot)}}(\Omega).$$

In addition, we assume that $\omega(x,r)$ satisfies the condition

$$\sup_{x\in\Omega} \|\omega(x,\cdot)\|_{L^{\theta(\cdot)}(0,\ell)} < \infty. \tag{12}$$

Theorem 1.6 Assume $p \in LH(\Omega)$ satisfy (1), $\ell = \operatorname{diam} \Omega < \infty$ and Let

$$1 < \theta_1^- \leqslant \theta_1(t) \leqslant \theta_1^+ < \infty,\ 0 < t < \ell,$$
$$1 < \theta_2^- \leqslant \theta_2(t) \leqslant \theta_2^+ < \infty,\ 0 < t < \ell.$$

if there exists $\delta > 0$ such that

$$\theta_1(t) \leqslant \theta_2(t)$$

for $t \in (0, \delta)$, θ_1 and ω_1 satisfy (12) and

$$\sup_{0<t<\delta, x\in\Omega} \int_0^t \omega_2(x,\xi)^{\theta_2(\xi)} \left(\int_t^\delta \frac{dr}{[r\omega_1(x,r)]^{[\widetilde{\theta}(\xi)]'}} \right)^{\frac{\theta_2(\xi)}{[\theta_1(\xi)]'}} d\xi < \infty,$$

where $\widetilde{\theta} = \inf_{s\in(\xi,\ell)} \theta_1(s)$, then G_α is bounded from $\mathscr{L}^{p(\cdot),\theta_1(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathscr{L}^{p(\cdot),\theta_2(\cdot),\omega_2(\cdot)}(\Omega)$.

2. Preliminaries

A nonnegative locally integrable function w belongs to A_p $(p > 1)$ if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where p' is the conjugate index of p i.e. $1/p + 1/p' = 1$.

Lemma 2.1. *[18]* The intrinsic square function G_α, the Lusin area integral S_β, the Littlewood-Paley function g, the continuous square functions S_ψ and g_ψ are bounded on $L^p(w)$ for $w \in A_p$ $(1 < p < \infty)$.

Lemma 2.2. *[20]* Let $0 < \alpha \leqslant 1$, $1 < p \leqslant \infty$ and $\omega \in A_p$. Then the commutators $[b, G_\alpha]$ and $[b, g^*_{\lambda,\alpha}]$ are bounded from $L^p(w)$ into itself whenever $b \in BMO$.

Lemma 2.3. *[19]* Let $0 < \alpha \leqslant 1$ and $\lambda > (3n + 2\alpha)/n$. Then for any $\sigma > 0$, there exists a constant $C > 0$ independent of f and $\sigma > 0$, such that

$$|\{x \in \mathbb{R}^n : g^*_{\lambda,\alpha}(f)(x) > \sigma\}| \leqslant \frac{C}{\sigma} \int_{\mathbb{R}^n} |f(x)|dx.$$

An important extrapolation theorem was introduced in [5].

Lemma 2.4. Let f and g be non-negative and measurable functions, assume that there exists a constant C such that

$$\int_\Omega f(x)^q w(x)dx \leqslant C \int_\Omega g(x)^q w(x)dx$$

holds for some $1 < q < \infty$ and for every $w \in A_q$. Then

$$\|f\|_{p(\cdot)} \leqslant C\|g\|_{p(\cdot)} \tag{13}$$

for $p(\cdot) \in LH(\Omega)$.

We can obtain the following conclusion by Lemma 2.1 and Lemma 2.4.

Theorem 2.1. (13) holds for the intrinsic square function $G_\alpha(f)$ $(0 < \alpha \leqslant 1)$, the Lusin area integral $S_\beta(f)$, the Littlewood-Paley function $g(f)$, the continuous square functions $S_\psi(f)$ and $g_\psi(f)$ for $f \in \mathscr{L}^{p(\cdot)}(\Omega)$ and $p(\cdot) \in LH(\Omega)$.

The Littlewood-Paley g^*_λ-function has been discussed in [5]. For commutators, we have the following theorem by using Lemma 2.2 and Lemma 2.4.

Theorem 2.2. (13) holds for $[b, G_\alpha](f)$ and $[b, g^*_{\lambda,\alpha}](f)$ for $f \in \mathscr{L}^{p(\cdot)}(\Omega)$ and $p(\cdot) \in LH(\Omega)$.

We will use the following estimate.

Lemma 2.5[9] Suppose that $1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty$, $p(\cdot) \in LH(\Omega)$ and

$$\sup_{x\in\Omega} \nu(x) < \infty, \ \inf_{x\in\Omega} [n + \nu(x)p(x)] > 0.$$

Then

$$\||x - \cdot|^{\nu(x)} \chi_{\widetilde{Q}(x,r)}\|_{p(\cdot)} \leqslant C r^{\nu(x)+\frac{n}{p(x)}}, \ x \in \Omega, \ 0 < r < \ell = \operatorname{diam} \Omega,$$

where C is independent of x and r.

[6, 7] assert that the constant C in Lemma 2 can be express as

$$C = C_0 \ell^{n(\frac{1}{p_-} - \frac{1}{p_+})},$$

where C_0 is independent of Ω.

Lemma 2.5. *[21]* Let $\theta(t)$ and $r(t)$ be measurable functions on $I = (0, \ell)$ such that

$$1 < \inf_{t\in(0,\ell)} \theta(t),\ \sup_{t\in(0,\ell)} r(t) < \infty,\ \theta(t) \leqslant r(t),\ t \in (0, \ell).$$

if

$$\sup_{0<t<l} \int_0^t v(\xi)^{r(\xi)} \left(\int_t^\ell \omega^{[\widetilde{\theta}(\xi)]'}(r)dr \right)^{\frac{r(\xi)}{[\theta(\xi)]'}} d\xi < \infty,$$

then $\overline{H}_{v,\omega}$ is bounded from $L^{\theta(\cdot)}(0, \ell)$ to $L^{r(\cdot)}(0, \ell)$, where $\widetilde{\theta} = \inf_{s\in(\xi,\ell)} \theta(s)$ and $\overline{H}_{v,\omega}$ is the weighted Hardy type operator

$$\overline{H}_{v,\omega}(f)(t) = v(t) \int_t^\ell f(\xi)\omega(\xi)d\xi.$$

Lemma 2.6. *[7]* For $p(\cdot) \in LH$ the embeddings

$$L^\infty(\Omega) \hookrightarrow \mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

holds if (12) is satisfied and there exist a $\delta \in (0, \ell)$ such that

$$\sup_{x\in\Omega} \|\omega(x, \cdot)\|_{L^{\theta(\cdot)}(\delta,\ell)} > 0.$$

Lemma 2.7. *[10]* Assume that $\omega(x, r)$ satisfy the conditions (8) and (9).
Then for $1 \leqslant q < p < \infty$, there is a constant $C_{p,q} > 0$ such that

$$\|M_q f\|_{p,\omega} \leqslant C_{p,q}\|f\|_{p,\omega} \quad for\ f \in L^{p,\omega},$$

where $(M(|f|)^q(x))^{1/q} = M_q f(x)$ and M is the Hardy-Littlewood maximal operator.

3. Proof of the Main Results

Proof of Theorem 1.2: By (4) and (5), it is enough to prove Theorem 1.1 for $G_\alpha(f)$. (i) Let Q be a cube of $\mathbb{R}^n$, with the center x_0 and side-length r. Decompose $f = f\chi_{2Q} + f\chi_{(2Q)^c}$ and denote $f\chi_{2Q}$ by f_1 and $f\chi_{(2Q)^c}$ by f_2, χ_{2Q} denotes the characteristic function of set $2Q$. Since $G_\alpha(f)$ is a sublinear, we have

$$\begin{aligned} & \left(\frac{1}{\omega(x_0, r)} \int_Q |G_\alpha(f)(x)|^p dx \right)^{1/p} \\ \leqslant & \left(\frac{1}{\omega(x_0, r)} \int_Q |G_\alpha(f_1)(x)|^p dx \right)^{1/p} + \left(\frac{1}{\omega(x_0, r)} \int_Q |G_\alpha(f_2)(x)|^p dx \right)^{1/p} \\ = & I_1 + I_2. \end{aligned}$$

Lemma 2.1 and (8) imply

$$\begin{aligned} I_1 &= \frac{1}{\omega(x_0, r)^{1/p}} \left(\int_{\mathbb{R}^n} |G_\alpha(f_1)(x)|^p dx \right)^{1/p} \\ &\leqslant C \frac{1}{\omega(x_0, 2r)^{1/p}} \left(\int_{2Q} |f(x)|^p dx \right)^{1/p} \\ &\leqslant C\|f\|_{p,\omega} \end{aligned}$$

For any $x \in Q$, $(y, t) \in \Gamma(x)$ and $z \in (2^{l+1}Q \setminus 2^l Q) \cap Q(y, t)$, $l \in Z^+$, then

$$2t \geqslant |x - y| + |y - z| \geqslant |x - z| \geqslant |z - x_0| - |x - x_0| \geqslant 2^{l-1}r.$$

By the Minkowski inequality and Hölder inequality, we have

$$\begin{aligned}
&|G_\alpha(f_2)(x)| \\
&= \left(\int_{\Gamma_1(x)} (A_\alpha(f_2)(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\
&= \left(\int_{\Gamma_1(x)} (\sup_{\phi \in \mathcal{C}_\alpha} |f * \phi_t(y)|)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\
&\leqslant C \bigg(\int_{2^{l-2}r}^{\infty} \int_{|x-y|<t} \bigg(t^{-n} \int_{(2Q)^c \cap \{z:|y-z|\leqslant t\}} |f(z)| dz \bigg)^2 \frac{dydt}{t^{n+1}} \bigg)^{1/2} \\
&\leqslant C \bigg(\int_{2^{l-2}r}^{\infty} \int_{|x-y|<t} \bigg(t^{-n} \sum_{l=1}^{\infty} \int_{(2^{l+1}Q \setminus 2^l Q)} |f(z)| dz \bigg)^2 \frac{dydt}{t^{n+1}} \bigg)^{1/2} \\
&\leqslant C \bigg(\sum_{l=1}^{\infty} \int_{(2^{l+1}Q \setminus 2^l Q)} |f(z)| dz \bigg) \bigg(\int_{2^{l-2}r}^{\infty} \frac{dt}{t^{2n+1}} \bigg)^{1/2} \\
&\leqslant C \sum_{l=1}^{\infty} \frac{1}{|2^{l+1}Q|} \int_{(2^{l+1}Q \setminus 2^l Q)} |f(z)| dz \\
&\leqslant C \sum_{l=1}^{\infty} \frac{\omega(x_0, 2^{l+1}r)^{1/p}}{|2^{l+1}Q|^{1/p}} \bigg(\frac{1}{\omega(x_0, 2^{l+1}r)} \int_{2^{l+1}Q} |f(z)|^p dz \bigg)^{1/p}.
\end{aligned}$$

It is easy to know that

$$\frac{\omega(x_0, 2^{l+1}r)}{|2^{l+1}Q|} \quad \text{is comparable to} \quad \int_{2^l r}^{2^{l+1}r} \frac{\omega(x_0,t)}{t^{n+1}} dt$$

by (8) and (9). So we get

$$\begin{aligned}
|G_\alpha(f_2)(x)|^p &\leqslant C \|f\|_{p,\omega}^p \sum_{l=1}^{\infty} \int_{2^l r}^{2^{l+1}r} \frac{\omega(x_0,t)}{t^{n+1}} dt \\
&\leqslant C \|f\|_{p,\omega}^p \int_r^{\infty} \frac{\omega(x_0,t)}{t^{n+1}} dt \\
&\leqslant C \|f\|_{p,\omega}^p \frac{\omega(x_0,r)}{r^n}.
\end{aligned}$$

Hence

$$I_2 \quad \leqslant \quad C\|f\|_{p,\omega}.$$

Combining the estimate for I_1 and I_2, we obtain the (10).
(ii) For $\sigma > 0$, by (3) and (10), we get (11). □
In order to prove 1.2 we need the following estimate.
Proposition 3.1. Let $0 < \theta \leqslant 1$. Assume that ω satisfies

$$1/C \leqslant \omega(x,t)/\omega(x,r) \leqslant C, \quad for \quad r \leqslant t \leqslant 2r$$

and

$$\int_r^{\infty} \frac{\omega(x,t)}{t^{n\theta+1}} dt \leqslant C \frac{\omega(x,r)}{r^{n\theta}}.$$

Then for $1 \leqslant p < \infty$ there is a constant $C > 0$ such that

$$\int_Q |G_N(f)(x)|^p dx \leqslant C\omega(Q) \|S_{\psi,\beta}(f)(x)\|_{p,\omega}^p$$

for $f \in L^{p,\omega}$, any cube Q, β and N large enough, where G_N is the grand maximal function .

The following important weight norm inequality is proved in [15]:

$$\int_{\mathbb{R}^n} |G_N(f)(x)|^p V(x)dx \leqslant C \int_{\mathbb{R}^n} (S_{\psi,\beta}(f)(x))^p \widetilde{M}V(x)dx. \tag{14}$$

That is for $0 < p < \infty$, there exits a "Maximal operator" $\widetilde{M}$ such that (14) is valid for $f \in L^p(\mathbb{R}^n)$, any nonnegative $V \in L^1_{loc}(\mathbb{R}^n)$, appropriate ψ and large enough β and N with a constant C which is independent V and f. For every p, we can take $\widetilde{M} = M^k$ $(k > 1)$; if $k = 1$, (14) holds for $p \in (0, 2)$.

Proof of proposition 3.1: Taking $V = \chi_Q$ in (14), we have

$$\int_{\mathbb{R}^n} |G_N(f)(x)|^p \chi_Q(x)dx \leqslant C \int_{\mathbb{R}^n} (S_{\psi,\beta}(f)(x))^p \widetilde{M}\chi_Q(x)dx.$$

Since χ_Q is the characteristic function of Q, $\widetilde{M}\chi_Q(x) \leqslant 1$. And for $x \in 2^{l+1}Q \setminus 2^lQ$, $\widetilde{M}\chi_Q(x)$ is comparable to 2^{-ln}, $l \in Z^+$.

Therefore

$$\begin{aligned}
&\int_Q |G_N(f)(x)|^p dx \\
&\leqslant C\left[\int_{2Q} (S_{\psi,\beta}(f)(x))^p dx + \sum_{l=1}^{\infty} \int_{(2^{l+1}Q\setminus 2^lQ)} 2^{-ln\theta}(S_{\psi,\beta}(f)(x))^p dx\right] \\
&\leqslant C\left[\omega(2Q) + \sum_{l=1}^{\infty} 2^{-ln\theta}\omega(2^{l+1}Q)\right] \|S_{\psi,\beta}(f)\|^p_{p,\omega} \\
&\leqslant Cr^{n\theta} \sum_{l=0}^{\infty} \frac{\omega(x_0, 2^l r)}{(2^l r)^{n\theta}} \|S_{\psi,\beta}(f)\|^p_{p,\omega}.
\end{aligned}$$

Since

$$\frac{\omega(x_0, 2^{l+1}r)}{|2^{l+1}Q|^{n\theta}} \text{ is comparable to } \int_{2^l r}^{2^{l+1}r} \frac{\omega(x_0, t)}{t^{n\theta+1}} dt,$$

we get

$$\begin{aligned}
\int_Q |G_N(f)(x)|^p dx &\leqslant Cr^{n\theta} \int_r^{\infty} \frac{\omega(x_0, t)}{t^{n\theta+1}} dt \|S_{\psi,\beta}(f)\|^p_{p,\omega} \\
&\leqslant C\omega(Q)\|S_{\psi,\beta}(f)\|^p_{p,\omega}.
\end{aligned}$$

□

Proof of 1.2: Wislon ([16, 17]) prove that there exists $\alpha \leqslant 1$ which depends on T such that for all $x \in \mathbb{R}^n$,

$$S_\psi(Tf)(x) \leqslant c(T, \psi, n)G_\alpha(f)(x).$$

By the same method yields

$$S_{\psi,\beta}(Tf)(x) \leqslant c(T, \psi, n, \beta)G_\alpha(f)(x), \tag{15}$$

for $\beta \geqslant 1$.

The Proposition 3.1 tells us

$$\|G_N(f)\|_{p,\varphi} \leqslant C\|S_{\psi,\beta}(f)\|_{p,\omega} \tag{16}$$

for large enough β and N.

It is well known that for all $x \in \mathbb{R}^n$ [22, pp. 67-68],

$$T^* f(x) \leqslant C(n, T)(G_N(Tf)(x) + Mf(x)).$$

Combining this with (15) and (16), we have

$$\begin{aligned}\|T^*f\|_{p,\omega} &\leqslant C\big(\|S_{\psi,\beta}(Tf)\|_{p,\omega}+\|Mf\|_{p,\omega}\big)\\ &\leqslant C\big(\|G_\alpha(f)\|_{p,\omega}+\|Mf\|_{p,\omega}\big).\end{aligned}$$

This estimate along with Theorem 1.1 and Lemma 2.7, (10) yields for T. □

Proof of Theorem 1.2 : (i) As before, we set $f=f_1+f_2$, $f_1=f\chi_{2Q}$. From the definition of $g^*_{\lambda,\alpha}$, we can easily deduce that

$$\begin{aligned}&\left(\frac{1}{\omega(x_0,r)}\int_Q(g^*_{\lambda,\alpha}(f)(x))^p dx\right)^{1/p}\\ \leqslant&\frac{C}{\omega(x_0,r)^{1/p}}\left[\left(\int_Q(G_\alpha(f)(x))^p dx\right)^{1/p}+\sum_{j=1}^\infty 2^{-j\lambda n/2}\left(\int_Q(G_{2^j,\alpha}(f)(x))^p dx\right)^{1/p}\right]\\ \leqslant&\frac{C}{\omega(x_0,r)^{1/p}}\left[\left(\int_Q(G_\alpha(f)(x))^p dx\right)^{1/p}+\sum_{j=1}^\infty 2^{-j\lambda n/2}\left(\int_Q(G_{2^j,\alpha}(f_1)(x))^p dx\right)^{1/p}\right.\\ &\left.+\sum_{j=1}^\infty 2^{-j\lambda n/2}\left(\int_Q(G_{2^j,\alpha}(f_2)(x))^p dx\right)^{1/p}\right]\\ =&I_0+\sum_{j=1}^\infty 2^{-j\lambda n/2}I_j^1+\sum_{j=1}^\infty 2^{-j\lambda n/2}I_j^2.\end{aligned}$$

By Theorem 1.1, we know that $I_0\leqslant C\|f\|_{p,\omega}$. We can get the following estimate for I_j^1 from [19], i.e.

$$I_j^1\leqslant C\|f\|_{p,\omega}(2^{jn/2}+2^{jn/p})\frac{\omega(x_0,2r)^{1/p}}{\omega(x_0,r)^{1/p}}.$$

We obtain

$$I_j^1\leqslant C\|f\|_{p,\varphi}(2^{jn/2}+2^{jn/p})$$

by (8).

The estimate of $I_j^{(2)}$ is similar to I_2,

$$\begin{aligned}&G_{2^j,\alpha}(f_2)(x)dx\\ \leqslant&C2^{3jn/2}\sum_{l=1}^\infty\frac{\omega(x_0,2^{l+1}r)^{1/p}}{|2^{l+1}Q|^{1/p}}\left(\frac{1}{\omega(x_0,2^{l+1}r)}\int_{2^{l+1}Q}|f(z)|^p dz\right)^{1/p}\\ \leqslant&C2^{3jn/2}\|f\|_{p,\omega}\left(\int_r^\infty\frac{\omega(x_0,t)}{t^{n+1}}dt\right)^{1/p}\\ \leqslant&C2^{3jn/2}\|f\|_{p,\omega}\frac{\omega(x_0,r)^{1/p}}{r^{n/p}},\end{aligned}$$

hence

$$I_j^{(2)}\leqslant C2^{3jn/2}\|f\|_{p,\omega}.$$

Therefore

$$\begin{aligned}&\left(\frac{1}{\omega(x_0,r)}\int_Q(g^*_{\lambda,\alpha}(f)(x))^p dx\right)^{1/p}\\ &\qquad\leqslant C\|f\|_{p,\omega}(1+\sum_{j=1}^\infty 2^{-j\lambda n/2}2^{3jn/2}+\sum_{j=1}^\infty 2^{-j\lambda n/2}2^{jn/p})\\ &\qquad\leqslant C\|f\|_{p,\omega}\end{aligned}$$

for $\lambda > 3 > 2/p$.
(ii)For $\sigma > 0$, by Lemma 2.3 and the estimate of $I_j^{(2)}$, the (11) is valid to $g^*_{\lambda,\alpha}$. □

Proof of Theorem 1.3 : Fix a cube Q of $\mathbb{R}^n$ whose center is x_0 and edges have length r. Decompose $f = f\chi_{2Q} + f\chi_{(2Q)^c}$ and denote $f\chi_{2Q}$ by f_1 and $f\chi_{(2Q)^c}$ by f_2.

$$\begin{aligned}&\left(\frac{1}{\omega(x_0,r)}\int_Q |[b,G_\alpha](f)(x)|^p dx\right)^{1/p}\\ \leqslant&\left(\frac{1}{\omega(x_0,r)}\int_Q |[b,G_\alpha](f_1)(x)|^p dx\right)^{1/p} + \left(\frac{1}{\varphi(x_0,r)}\int_Q |[b,G_\alpha](f_2)(x)|^p dx\right)^{1/p}\\ =&II_1 + II_2.\end{aligned}$$

By Lemma 2.2 we get

$$\begin{aligned}II_1 &\leqslant C\frac{1}{\omega(x_0,r)^{1/p}}\left(\int_{\mathbb{R}^n} |[b,G_\alpha](f_1)(x)|^p dx\right)^{1/p}\\ &\leqslant C\frac{\|b\|_{BMO}}{\omega(x_0,2r)^{1/p}}\left(\int_{2Q} |f(x)|^p dx\right)^{1/p}\\ &\leqslant C\|f\|_{p,\omega}\|b\|_{BMO}.\end{aligned}$$

Now we estimate II_2. For any $x \in Q$, $(y,t) \in \Gamma(x)$, since $|b(x) - b(z)| \leqslant |b(x) - b_Q| + |b(z) - b_Q|$, we have

$$\begin{aligned}|[b,G_\alpha](f_2)(x)| &\leqslant |b(x)-b(z)|G_\alpha(f_2)(x) + \left(\int_\Gamma \sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(z)-b_Q]\phi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &= II_{11} + II_{21}.\end{aligned}$$

In the proof of Theorem 1.1, we get that for any $x \in Q$,

$$|G_\alpha(f_2)(x)|^p \leqslant C\|f\|^p_{p,\omega}\frac{\omega(x_0,r)}{r^n}.$$

And by the (6), we obtain

$$\begin{aligned}\frac{1}{\omega(x_0,r)}\int_Q II^p_{11}dx &\leqslant C\|f\|^p_{p,\omega}\frac{1}{|Q|}\int_Q |b(x)-b_Q|^p dx\\ &\leqslant C\|f\|^p_{p,\omega}\|b\|^p_{BMO}.\end{aligned}$$

Next we deal with II_{21}. It is similar to $|G_\alpha(f_2)(x)|$,

$$\begin{aligned}II_{21} \leqslant&\left(\int_{\Gamma_1(x)}\sup_{\phi\in\mathcal{C}_\alpha}\left(\int_{\mathbb{R}^n}|(b(z)-b_Q)\phi_t(y-z)f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ \leqslant&C\left(\int_{\Gamma_1(x)}\left(t^{-n}\sum_{l=1}^{\infty}\int_{\{2^{l+1}Q\backslash 2^lQ\}\cap\{z:|y-z|\leqslant t\}}|b(z)-b_{2^{l+1}Q}||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\\ &+C\left(\int_{\Gamma_1(x)}\left(t^{-n}\sum_{l=1}^{\infty}\int_{\{2^{l+1}Q\backslash 2^lQ\}\cap\{z:|y-z|\leqslant t\}}|b_{2^{l+1}Q}-b_Q||f(z)|dz\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\end{aligned}$$

$$\begin{aligned}
&\leqslant C\sum_{l=1}^{\infty}\frac{1}{|2^{l+1}Q|}\int_{(2^{l+1}Q\backslash 2^{l}Q)}|b(z)-b_{2^{l+1}Q}||f(z)|dz\\
&\quad+C\sum_{l=1}^{\infty}\frac{|b_{2^{l+1}Q}-b_{Q}|}{|2^{l+1}Q|}\int_{(2^{l+1}Q\backslash 2^{l}Q)}|f(z)|dz\\
&\leqslant\|f\|_{p,\omega}\left(\int_{r}^{\infty}\frac{\omega(x_0,t)}{t^{n+1}}dt\right)^{1/p}\left(\frac{1}{|Q|}\int_{Q}|b(x)-b_{Q}|^{p'}dx\right)^{1/p'}\\
&\quad+C\|f\|_{p,\omega}\sum_{l=1}^{\infty}\frac{|b_{2^{l+1}Q}-b_{Q}|\omega(x_0,2^{l+1}r)^{1/p}}{|2^{l+1}Q|^{1/p}}.
\end{aligned}$$

Applying (6), (7) and $\omega(x_0,2^{l+1}r)\sim\omega(x_0,r)$, we have

$$\begin{aligned}
II_{21} &\leqslant C\|f\|_{p,\omega}\|b\|_{BMO}\frac{\omega(x_0,r)^{1/p}}{r^{n/p}}+C\|f\|_{p,\omega}\|b\|_{BMO}\frac{\omega(x_0,r)^{1/p}}{r^{n/p}}\sum_{l=1}^{\infty}\frac{l+1}{2^{(l+1)n/p}}\\
&\leqslant C'\|f\|_{p,\omega}\|b\|_{BMO}\frac{\omega(x_0,r)^{1/p}}{r^{n/p}}.
\end{aligned}$$

Hence

$$\frac{1}{\omega(x_0,r)}\int\limits_{Q}II_{21}^{p}dx\leqslant C\|f\|_{p,\omega}^{p}\|b\|_{BMO}^{p}.$$

Combining II_{11} and II_{21}, so

$$II_2\leqslant C\|f\|_{p,\omega}\|b\|_{BMO}.$$

Thus (9) holds for $[b,G_\alpha]$. □

Proof of Theorem 1.4 : By Lemma 2.2, we can use the same arguments in the proof of Theorem 1.2 (i) and Theorem 1.3 to get the conclusion of Theorem 1.4. □

We use the following estimate to show the Theorem 1.5 and Theorem .

Proposition 3.2 Let Ω be bounded, $\ell=\operatorname{diam}\Omega$ and $p\in HL(\Omega)$ satisfy (1). Then

$$\|G_\alpha f\|_{L^{p(\cdot)}(\widetilde{Q}(x,t))}\leqslant Ct^{\frac{n}{p(x)}}\int_{t}^{\ell}r^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}dr,\ 0<t<\delta$$

for $f\in L^{p(\cdot)}(\Omega)$, where C is independent of f, $x\in\Omega$ and t.

Proof Proposition 3.2: Let Q be a cube of $\mathbb{R}^n$, the center is x_0 and side length is t. Decompose $f=f\chi_{2\widetilde{Q}}+f\chi_{(2\widetilde{Q})^c}$ and denote $f\chi_{2\widetilde{Q}}$ by f_1 and $f\chi_{(2\widetilde{Q})^c}$ by f_2, $\chi_{2\widetilde{Q}}$ denotes the characteristic function of set $2\widetilde{Q}$. Since G_α is a sublinear, we have

$$\|G_\alpha(f)\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,t))}\leqslant\|G_\alpha(f_1)\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,t))}+\|G_\alpha(f_2)\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,t))}.$$

We get

$$\|G_\alpha(f_1)\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,t))}\leqslant\|G_\alpha(f_1)\|_{L^{p(\cdot)}(\Omega)}\leqslant C\|f_1\|_{L^{p(\cdot)}(\Omega)}=C\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,2t))}$$

by Theorem 2.1.

Then we easily obtain

$$\begin{aligned}
\|G_\alpha(f)\|_{L^{p(\cdot)}(\widetilde{Q}(x,t))} &\leqslant Ct^{\frac{n}{p(x)}}\int_{2t}^{\ell}r^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}dr\\
&\leqslant Ct^{\frac{n}{p(x)}}\int_{t}^{\ell}r^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}dr.
\end{aligned}$$

Next we estimate $G_\alpha(f_2)$. It is similar to the second estimate in proof (i) of Theorem 1.1. We have

$$\begin{aligned}|G_\alpha(f_2)(x)| &\leqslant C\sum_{l=1}^{\infty}\frac{1}{|2^{l+1}Q|}\int_{\Omega\cap(2^{l+1}Q\setminus 2^lQ)}|f(z)|dz\\ &\leqslant C\int_{\Omega\setminus\widetilde{Q}(x_0,2t)}\frac{|f(z)|}{|x_0-z|^n}dz.\end{aligned}$$

Choosing $\varepsilon > \frac{n}{p_-}$, by the Hölder inequality of variable exponent and Lemma 2 we have

$$\begin{aligned}&\int_{\Omega\setminus\widetilde{Q}(x_0,t)}\frac{|f(z)|}{|x_0-z|^n}dz\\ &\leqslant\varepsilon\int_{\Omega\setminus\widetilde{Q}(x_0,t)}\frac{|f(z)|}{|x_0-z|^{n-\varepsilon}}\left(\int_{|x-y|}^{\ell}\frac{1}{s^{\varepsilon+1}}ds\right)dz\\ &\leqslant\varepsilon\int_t^{\ell}\frac{1}{s^{\varepsilon+1}}ds\int_{\Omega\cap\{y:|x_0-z|\leqslant s\}}\frac{|f(z)|}{|x_0-z|^{n-\varepsilon}}dz\\ &\leqslant C\int_t^{\ell}\frac{1}{s^{\varepsilon+1}}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,s))}\||x_0-z|^{-n+\varepsilon}\|_{L^{p'(\cdot)}(\widetilde{Q}(x_0,s))}ds\\ &\leqslant C\int_t^{\ell}s^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,s))}ds.\end{aligned}$$

Hence

$$\begin{aligned}|G_\alpha(f_2)(x)| &\leqslant C\int_{2t}^{\ell}s^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,s))}ds\\ &\leqslant C\int_{t}^{\ell}s^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,s))}ds.\end{aligned}$$

By Lemma 2 and the Hölder inequality of variable exponent again, we get

$$\begin{aligned}\|G_\alpha(f_2)\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,t))} &\leqslant C\int_{t}^{\ell}s^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,s))}ds\|1\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,t))}\\ &\leqslant Ct^{\frac{n}{p(x)}}\int_{t}^{\ell}s^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x_0,s))}ds. \qquad\square\end{aligned}$$

Proof of Theorem 1.5 :

$$\|G_\alpha(f)\|_{p(\cdot),\omega_2}=\sup_{x\in\Omega,r>0}\frac{r^{-\frac{n}{p(x)}}}{\omega_2(x,r)}\|G_\alpha(f)\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}.$$

We just consider $r\in(0,\delta)$ under the assumption of (2). Applying Proposition 3.2, we have

$$\begin{aligned}&\sup_{x\in\Omega,r\in(0,\delta)}\frac{r^{-\frac{n}{p(x)}}}{\omega_2(x,r)}\|G_\alpha(f)\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}\\ &\leqslant C\sup_{x\in\Omega,r>0}\frac{1}{\omega_2(x,r)}\int_r^{\ell}s^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x,s))}ds\\ &\leqslant C\|f\|_{p(\cdot),\omega_1}\frac{1}{\omega_2(x,r)}\int_r^{\ell}\frac{\omega_1(x,s)}{s}ds\\ &\leqslant C\|f\|_{p(\cdot),\omega_1} \qquad\square\end{aligned}$$

Proof of Theorem :

$$\begin{aligned}\|G_\alpha(f)\|_{\mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)} &= \sup_{x\in\Omega}\left\|\frac{\omega(x,r)}{r^{\frac{n}{p(x)}}}\|G_\alpha(f)\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}\right\|_{L^{\theta(\cdot)}(0,\ell)} \\ &\leqslant \sup_{x\in\Omega}\left\|\frac{\omega(x,r)}{r^{\frac{n}{p(x)}}}\|G_\alpha(f)\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}\right\|_{L^{\theta(\cdot)}(0,\delta)} + \sup_{x\in\Omega}\left\|\frac{\omega(x,r)}{r^{\frac{n}{p(x)}}}\|G_\alpha(f)\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}\right\|_{L^{\theta(\cdot)}(\delta,\ell)} \\ &= I_1 + I_2\end{aligned}$$

The estimate of I_2 can directly follow from the Lemma 2.6.

$$\begin{aligned}I_2 &\leqslant C\|G_\alpha(f)\|_{\mathscr{L}^{p(\cdot)}(\Omega)}\|\omega_2(x,\cdot)\|_{L^{\theta(\cdot)}(\delta,\ell)} \\ &\leqslant C\|G_\alpha(f)\|_{\mathscr{L}^{p(\cdot)}(\Omega)} \\ &\leqslant C\|f\|_{\mathscr{L}^{p(\cdot)}(\Omega)} \leqslant C\|f\|_{\mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)}.\end{aligned}$$

For $t \in (0,\delta)$, We find

$$I_1 \leqslant C\left\|\omega_2(x,t)\int_t^\ell r^{-\frac{n}{p(x)}-1}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}dr\right\|_{L^{\theta(\cdot)}(\delta,\ell)}.$$

We splitting the integral with respect to r into two integrals over $(0,\delta)$ and (δ,ℓ). The estimate of integral over (δ,ℓ) is obtained as I_2 and the estimate of integral over $(0,\delta)$ is used by Lemma 2.5. In addition, by the Lemma 2.6 we get

$$\begin{aligned}I_1 &\leqslant C\|f\|_{\mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)} + \sup_{x\in\Omega}\left\|\frac{\omega(x,r)}{r^{\frac{n}{p(x)}}}\|f\|_{L^{p(\cdot)}(\widetilde{Q}(x,r))}\right\|_{L^{\theta(\cdot)}(0,\delta)} \\ &\leqslant C\|f\|_{\mathscr{L}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)}.\end{aligned}$$

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