American Journal of Applied Mathematics
2022; 10(6): 240-243
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20221006.13
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

Generalisation of Euler's Identity in the Form of K-Hypergeometric Functions

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To cite this article:
Nafis Ahmad, Mohd Sadiq Khan, Mohammad Imran Aziz. Generalisation of Euler's Identity in the Form of K-Hypergeometric Functions. *American Journal of Applied Mathematics*. Vol. 10, No. 6, 2022, pp. 240-243. doi: 10.11648/j.ajam.20221006.13

Received: October 27, 2022; **Accepted**: November 28, 2022; **Published**: December 29, 2022

Abstract: In this paper, we present a generalization of Euler's identity associated to usual hypergeometric function in the form of an identity associated with the k-hypergeometric function. The second-order homogeneous k-hypergeometric differential equation $kz(1-kz)\frac{d^2y}{dz^2}+[c-(k+a+b)kz]\frac{dy}{dz}-aby=0$, by Frobenious method yields a pair $\{y_1(z), y_2(z)\}$ of linearly independent solutions in the form of k-hypergeometric function $2F1, \mathrm{k}$ define as k-hypergeometric power series is convergent in the region $=\{z: |z|<1/k\}$. Here with suitable substitution to $y(z)$, we deduce two other forms of solutions of this equation near the singularity $z=0$. Using the dependency of these forms on $\{\, y_1(z), y_2(z)\,\}$, we obtain the generalized Euler's identity in the form of k-hypergeometric function and a new k-hypergeometric transformation formula. Our generalization pertains to the case when the generalized Euler's identity reduced to the classical Euler's identity. In the ultimate section of the paper, we obtain another reduction formula for a particular product difference of k-hypergeometric functions.

Keywords: K-Hypergeometric Equations, Frobenious Method, K-Hypergeometric Series Solutions, Regular Singular Point

1. Introduction

The Gauss hypergeometric equation which is expressed by

$$z(1-z)\frac{d^2y}{dz^2}+[c-(1+a+b)z]\frac{dy}{dz}-aby=0 \qquad (1)$$

with 3 regular singularities $\{0,1,\infty\}$ has been extensively studied by various authors including Coddington [1], Campos [2], Gasper [4], RainVille [9], Slater [10], Whittaker [12].

A hypergeometric series solution $2F1a,b;c;z]=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$
of (1) can be derived by the Frobenius method is convergent in the region.

$R=\{z: |z|<1\}, when\ c\notin z, z^{1-c}2\mathrm{F}1[\mathrm{a}-\mathrm{c}+1, \mathrm{b}-\mathrm{c}+1; 2-\mathrm{c}; \mathrm{z}]$ is also a linearly independent solution of (1) and convergent in the region R.

The k-hypergeometric function has been proposed as a generalization of the Gauss hypergeometric function.

The k-hypergeometric equation is given by

$$kz(1-kz)\frac{d^2y}{dz^2}+[c-(k+a+b)kz]\frac{dy}{dz}-aby=0 \quad (2)$$

And using Frobenious method, we have

$$y_1(z)=2F1,\mathrm{k}[(\mathrm{a},\mathrm{k}),(b,k);(c,k);z)=\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}}\frac{z^n}{n!}c\notin N_0$$

is a solution of (2). This is the k-hypergeometric function. We can showthat $z^{1-\frac{c}{k}}y_1(z)$ is again a solution of (2) but with different coefficients.

Using this we can prove that

$$y_2(z)=\ z^{1-\frac{c}{k}}2F1,\mathrm{k}[(\mathrm{a}+\mathrm{k}-\mathrm{c},\mathrm{k}),(b+k-c,k);(2k-c,k);z)]$$

is second solution for $c-2k\notin N_0$. This gives two linearly independent solutions of (2) $c\notin Z$.

Here one complete solution of (2) is

$$y(z) = A\, 2F1, \mathrm{k}[(a,k),(b,k),(c,k);z] + Bz^{1-\frac{c}{k}}\, 2F1, \mathrm{k}[(\mathrm{a}+\mathrm{k}-\mathrm{c},\mathrm{k}),(b+k-c,k);(2k-c,k);z)]$$

for $|z| < \frac{1}{k}$ $c \notin Z$ where A and B are constant. One may refer to the works of Krasniqi [5], Mubeen [6-8], Shengfeng [11], Abdalla [13], Ali [14] and Yil-mazer [15].

In the present paper by substitution, we compute two more solutions of the equation (2) near the singularity z=0, but equation (2), being of second order, can have at most two linearly independent solutions. Now by expressing the additional solutions in the form of complete solutions, we have deduce generalised form of Euler identity.

2. Preliminaries

In this section, we briefly review some basic definitions and facts concerning the Fuchsion type differential equation and k-hypergeometric series. Some surveys and literature for k-hypergeometric series and the k- hypergeometric differential equation can be found in Diaz et al. [3], Krasniqi [5], Mubeen [6-8], Shengfeng [11], Abdalla [13], Ali [14] and Yilmazer [15].

Definition 1: Assume that p(z) and q(z) be two complex valued functions. Let a second order differential equation in standard form:

$$\frac{d^2y}{dz^2} + p(z)\frac{dy}{dz} + q(z)y(z) = 0 \qquad (3)$$

Then the method about finding an infinite series solution of equation (3) is called the Frobenius method.

Definition 2: If$z = z_0$ is a singularity of (3), then $z = z_0$ is a regular singularity of (3) if and only if $(z-z_0)p(z), (z-z_0)^2 q(z)$ are analytic in $\{z: |z-z_0| < R\}$, where R is a positive real number.

Definition 3:

The Pochhammer k-symbol $(\sigma)_{n,k}$ is defined by:

$$(\sigma)_{n,k} = \sigma(\sigma+k)(\sigma+2k)\ldots(\sigma+(n-1)k), \sigma \neq 0 \qquad (4)$$

$(\sigma)_{0,k} = 1$ where $k > 0$

Definition 4: The k-hypergeometric series with three parameters a, b, and c is defined as:

$$2F1, \mathrm{k}[(\mathrm{a},\mathrm{k}),(b,k);(c,k);z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!} \qquad (5)$$

3. Theorem

The homogeneous k-hypergeometric differential equation (2) have a general solution in the form

$$y(z) = A\, 2F1, \mathrm{k}[(a,k),(b,k);(c,k);z] + B\, z^{1-\frac{c}{k}}\, 2F1, \mathrm{k}[(\mathrm{a}+\mathrm{k}-\mathrm{c},\mathrm{k}),(b+k-c,k);(2k-c,k);z)]$$

for $|z| < \frac{1}{k}, k \in R^+, a,b,c \in R$ and 2k-c neither zero nor negative integer.

Proof: Let us assume that

$$y = \sum_{r=0}^{\infty} u_r z^{l+r}\ (where\ u_0 \neq 0)$$

Is any solution of the equation (2). Then by direct differentiation of the series, wehave

$$kz(1-kz)\sum_{r=0}^{\infty}(l+r)(l+r-1)u_r z^{l+r-2} + [c-(a+b+k)kz]\sum_{r=0}^{\infty}(l+r)u_r z^{l+r-1} - ab\sum_{r=0}^{\infty} u_r z^{l+r} = 0 \qquad (6)$$

$$\sum_{r=0}^{\infty} k(l+r)(l+r-1)u_r z^{l+r-1} + \sum_{r=0}^{\infty} c(l+r)u_r z^{l+r-1} - \sum_{r=0}^{\infty} k^2(l+r)(l+r-1)u_r z^{l+r} - \sum_{r=0}^{\infty} k(a+b+c)(l+r)u_r z^{l+r} - ab\sum_{r=0}^{\infty} u_r z^{l+r} = 0 \qquad (7)$$

$$kz^l\sum_{r=1}^{\infty}(l+r)(l+r-1)u_r z^{r-1} + cz^l\sum_{r=1}^{\infty}(l+r)u_r z^{r-1} - k^2 z^l\sum_{r=1}^{\infty}(l+r-1)(l+r-2)u_{r-1}z^{r-1} - k(a+b+c)z^l\sum_{r=1}^{\infty}(l+r-1)u_{r-1}z^{r-1} - ab\, z^l\sum_{r=1}^{\infty} u_{r-1}z^{r-1} + u_0 l(k(l-1)+c)z^{l-1} = 0 \qquad (8)$$

Hence by comparing the coefficient on both sides of above equation, we must have as the indicial equation

$$l[k(l-1)+c] = 0 \qquad (9)$$

and the difference equation

$$(l+r+1)[k(l+r)+c]u_{r+1} = [k(l+r)+a][k(l+r)+b]u_r \qquad (10)$$

for $r = 0, 1, 2, \ldots$.

On solving the indicial equation (9), we have $l = 0$ and $l = 1 - \frac{c}{k}$

Case I: $l = 0$, From equation (10), we have the solution of equation (2):

$$y_1(z) = u_0 2F1,\mathrm{k}[(a,k),(b,k),(c,k);z] \tag{11}$$

Provided that c is neither zero nor a negative integer.

Case II: $l = 1 - \frac{c}{k}$. From equation (10), we have a difference equation

$$(r+1)(2k-c+kr)u_{r+1} = (a+k-c+kr)(b+k-c+kr)u_r \tag{12}$$

and equation (12) gives a second solution

$$y_2(z) = u_0 z^{1-\frac{c}{k}} 2F1,\mathrm{k}[(\mathrm{a+k-c,k}),(b+k-c,k);(2k-c,k);z)] \tag{13}$$

Provided that c is not a positive integer $\geq 2k$

Hence one complete solution of equation (2) is

$$y(z) = A\, 2F1,\mathrm{k}[(a,k),(b,k);(c,k);z] + B\, z^{1-c/k}\, 2F1,\mathrm{k}[(\mathrm{a+k-c,k}),(b+k-c,k);(2k-c,k);z) \tag{14}$$

Where $|z| < \frac{1}{k}$ and A, B are constant.

4. Identities with K-hypergeometric Functions

The following identities with the k-hypergeometric functions hold.

$$2F1,\mathrm{k}[(a,k),(b,k);(c,k);z] = (\tfrac{1}{k} - z)^{\frac{1}{k}(c-a-b)} 2F1,\mathrm{k}[(\mathrm{c-a,k}),(c-b,k);(c,k);z)]$$

and

$$2F1,\mathrm{k}[(\mathrm{a+k-c,k}),(b+k-c,k);(2k-c,k);z)] = (\tfrac{1}{k} - z)^{\frac{1}{k}(c-a-b)} 2F1,\mathrm{k}[(\mathrm{k-a,k}),(k-b,k);(2k-c,k);z)]$$

Proof: Let

$$y(z) = \left(\tfrac{1}{k} - z\right)^s w(z) \tag{15}$$

$$y' = \frac{1}{k^s}[(1-kz)^s \frac{dw}{dz} - sk(1-kz)^{s-1} w]$$

$$y'' = \frac{1}{k^s}[(1-kz)^s \frac{d^2w}{dz^2} - 2sk(1-kz)^{s-1}\frac{dw}{dz} + sk^2(s-1)(1-kz)^{s-2} w]$$

On plunging everything into the equation (2) yields the equation (16)

$$kz(1-kz)w'' + [c-(a+b+k+2sk)kz]w' + \left[\frac{s(s-1)k^3z - sk\{c-(a+b+k)kz\}}{(1-kz)} - ab\right] w = 0 \tag{16}$$

This equation is also of k-hypergeometric type if $(1-kz)$ exactly divides $s(s-1)k^3z - sk\{c-(a+b+k)kz\}$, that is, if either $s = 0$ or $s = \frac{1}{k}(c-a-b)$.

When $s = 0$ the two solutions (11) and (13) are given; but when$s = \frac{1}{k}(c-a-b)$, then two new solutions are given, valid in the region $|z| < \frac{1}{k}$.

These are

$$y_3(z) = (\tfrac{1}{k} - z)^{\frac{1}{k}(c-a-b)} 2F1,\mathrm{k}[(\mathrm{c-a,k}),(c-b,k);(c,k);z)] \tag{17}$$

and

$$y_4(z) = z^{1-\frac{c}{k}}(\tfrac{1}{k} - z)^{\frac{1}{k}(c-a-b)} 2F1,\mathrm{k}[(\mathrm{k-a,k}),(k-b,k);(2k-c,k);z)] \tag{18}$$

Since there exist only two linearly independent solution of (2), so we have

$$y_3 = \alpha_1 y_1 + \alpha_2 y_2 \text{ and } y_4 = \beta_1 y_1 + \beta_2 y_2$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are some constants.

$$(\frac{1}{k}-z)^{\frac{1}{k}(c-a-b)}\,2F1,\mathrm{k}[(\mathrm{c}-\mathrm{a},\mathrm{k}),(c-b,k);(c,k);z)]=\alpha_1\,2F1,\mathrm{k}[(a,k),(b,k);(c,k);z]+\alpha_2\,z^{1-\frac{c}{k}}\,2F1,\mathrm{k}[(\mathrm{a}+\mathrm{k}-\mathrm{c},\mathrm{k}),(b+k-c,k);(2k-c,k);z)],\mathrm{c}\notin\mathrm{Z}$$

Now setting $k=1$ and using the fact that the left-hand side of the resulting equation can be expanded in the integral power of z, but $z^{1-c}(\mathrm{c}\notin\mathrm{Z})$ cannot. Hence α_2 must be zero. Again for $k=1$ and $z=0$, we get $\alpha_{1,}=1$.

Hence $y_1(z)=y_3(z)$

$$2F1,\mathrm{k}[(a,k),(b,k);(c,k);z]=(\frac{1}{k}-z)^{\frac{1}{k}(c-a-b)}2F1,\mathrm{k}[(\mathrm{c}-\mathrm{a},\mathrm{k}),(c-b,k);(c,k);z)] \tag{19}$$

This is the generalised form of Euler's identity.

Similarly, we can show that $y_2(z)=y_4(z)$, that is

$$2F1,\mathrm{k}[(\mathrm{a}+\mathrm{k}-\mathrm{c},\mathrm{k}),(b+k-c,k);(2k-c,k);z)]=(\frac{1}{k}-z)^{\frac{1}{k}(c-a-b)}2F1,\mathrm{k}[(\mathrm{k}-\mathrm{a},\mathrm{k}),(k-b,k);(2k-c,k);z)] \tag{20}$$

These are the two identities with k-hypergeometric functions.

If we take $k=1$ in (19), we have

$$2F1[a,b;c;z]=(1-z)^{(c-a-b)}2F1[\mathrm{c}-\mathrm{a},\mathrm{c}-\mathrm{b};\mathrm{c};\mathrm{z}] \tag{21}$$

Which is known as Euler identity for usual hypergeometric functions.

Now identities (19) and (20) immediately leads to the reduction formula

$$2F1,\mathrm{k}[(a,k),(b,k);(c,k);z]2F1,\mathrm{k}[(\mathrm{k}-\mathrm{a},\mathrm{k}),(k-b,k);(2k-c,k);z)]-2F1,\mathrm{k}[(\mathrm{c}-\mathrm{a},\mathrm{k}),(c-b,k);(c,k);z)]\,2F1,\mathrm{k}[(\mathrm{a}-\mathrm{c}+\mathrm{k},\mathrm{k}),(b-c+k,k);(2k-c,k);z)]=0 \tag{22}$$

5. Conclusion

In this paper, we investigate the two independent Frobenious solutions of second order k- hypergeometric differential equations near the singularity z=0, and two other dependent solutions by making some suitable substitution. We also have suggested two identities for the k-hypergeometric functions, which can be used in the study of summation, transformation and reduction formulae related to k-hypergeometric functions. If we take k=1, then the suggested identity reduces to the Euler's identity for usual hypergeometric function. It would be interesting to have more research about this case.

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