American Journal of Applied Mathematics
2023; 1(1): 7-16
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20231101.12
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

Finite-Time State Estimation of Switched Neural Networks with Both Time-Varying Delays and Leakage Delay

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To cite this article:

Fangjing Zheng, Zhifeng Lu. Finite-Time State Estimation of Switched Neural Networks with Both Time-Varying Delays and Leakage Delay. *American Journal of Applied Mathematics*. Vol. 11, No. 1, 2023, pp. 7-16. doi: 10.11648/j.ajam.20231101.12

Received: December 25, 2022; **Accepted**: January 19, 2023; **Published**: February 6, 2023

Abstract: In this thesis, we deal with the issues of the finite-time state estimation (FTSE) for a set of switched neural networks (SNNs), in which the hybrid effects of time-varying delays and leakage delay are taken into consideration. Therefore, the model of SNNs under discussion is quite comprehensive and more practical. In the light of an applicable piecewise Lyapunov-Krasovskii (L-K) functional which has double integral terms, some novel sufficient criteria are put forward with the average dwell time (ADT) technique, so that the estimation error system is finite-time boundedness (FTB). It is crucial to notice that the estimation results in our work are time-delay dependent, which depend on the leakage delay as well as the upper bound of the time-varying delays. The results show that the unknown gain matrix of the state estimator is achieved by solving a series of linear matrix inequalities (LMIs), which can be effortlessly tested with the MATLAB Toolbox. Moreover, by combining with free weight matrix method in the proof process, the results we obtained do not require the differentiability of time-varying delays any more, which is less conservative than some existing results. Finally, an example is performed with its numerical simulations to corroborate the efficiency of the theoretical results.

Keywords: Finite-Time State Estimation, Switched Neural Networks, Time-Varying Delays, Leakage Delay

1. Introduction

Neural networks are the nonlinear macroscale adaptive dynamic system composed of many processing units. A lot of engineering applications of neural networks have been founded, for instance, image reparation, automation control and system identification [1–3]. Since the practical setting is fully complex and changeable, the structure of the networks may change unpredictably. Therefore, the idea of switching system is developed to solve the problem that the structure of neural networks may change due to volatile external factors. In recent years, SNNs have been extensively used in the field of high-speed signal processing, gene selection for DNA microarray analysis and artificial intelligence [4–6]. In addition, due to the limited switching speed of amplifiers or the limited propagation time of signals in biological networks, there are always inevitable delays in actual SNNs, which may give rise to undesirable dynamic behaviors, for instance, chaos, oscillation and multiple cycles [7-10]. Recently, the dynamic behaviors of delayed SNNs have attracted a lot of research efforts [11-13].

In practical engineering, the state of the system may not be completely determined by the output of the networks. It is often necessary to design observers to estimate each neuron with the output of the neural networks and achieve certain design goals by estimating the state of the neurons. At present, there are many results about state estimation of SNNs. For example, under the available output measurements and the multiple L-K functional technique, the exponential state estimation of switched interval neural networks with mixed delays has been carried out [14]. Based on a proper L-K functional, the author has derived a new sufficient condition making the estimation error system of SNNs be exponentially stable [15]. The state estimation of discrete-time SNNs with mode-dependent time-varying delays has been investigated in H_∞ sense [16].

What should be noticed is that the design results of the above state estimator for SNNs are defined in an infinite time range. However, in practice, calculating the value of the system state beforehand in order to remain within the specified range during the relevant time interval provides better transient performance and in turn prevents saturation from occurring [17-19]. Although some significant work has been done for the FTSE of non-switched delayed neural networks [20–22], few studies focused on the corresponding research for SNNs. Additionally, leakage delay existing in a negative feedback term is hard to process since it has quick tendency to destroy the system performance [23–25]. However, the leakage delay is often ignored in most past modeling of SNNs [26]. Therefore, it is of great theoretical significance to investigate the design of finite-time estimator for SNNs when the combined effects of time-varying delays and leakage delay are taken fully into account. For all we know, there is no relevant results on this issue yet, which motivates our study.

In this thesis, we explore the issue of FTSE of the SNNs with time-varying delays and leakage delay. By constructing the appropriate piecewise L-K functional, employing ADT method and free-weighting matrix technique, some sufficient conditions are educed to make sure the FTB of the corresponding estimation error system. The existence and the characterization of the desired estimator can be achieved based on a series of LMIs. The results we obtained rely on both time-varying delays and leakage delay. Moreover, our results do not require the time delays to be differentiable, which cannot be applicable to the systems with inestimable or unknown time delays. The remaining part is arranged as follows. The model description and pre-knowledge are introduced in Section 2. The primary theorems are obtained in Section 3. An example is performed with its numerical simulations to corroborate the efficiency of the developed theoretical results in Section 4. The summary of our work is given in Section 5.

Notations. Let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$ be the set of real numbers, the set of all positive real numbers, the n -dimensional real space, the $n\times m$ -dimensional real space, respectively. $A<0$ ($A>0$) denotes that matrix A is a symmetric and negative definite (positive definite) matrix. The notation A^{-1} and A^T represent the inverse and transpose of matrix A, respectively. If A and B are symmetric matrices, $A>B$ means that $A-B$ is a positive definite matrix. $\lambda_{max}(A)$ ($\lambda_{min}(A)$) stands for the maximum (minimum) eigenvalue of matrix A. Unless otherwise specified, I is the identity matrix with appropriate dimensions. $\Lambda=\{1,2,\dots,n\}$ and $\mathbb{N}=\{1,2,\dots,N\}$ is a finite set formed by the set of natural numbers. $a\vee b$ denotes the maximum value of a and b. $C^1(U,V)=\{\overline{\varphi}:U\to V$ is continuously differentiable$\}$ for any interval $U\subseteq\mathbb{R}$ and set $V\subseteq\mathbb{R}^k (1\le k\le n)$. Notation $\star$ is the symmetric block in a matrix.

2. Preliminaries

This paper considers the SNNs with time-varying delays and leakage delay in the following form

$$\begin{cases}\dot{x}(t)=-A_{\sigma(t)}x(t-\delta)+B_{\sigma(t)}\overline{f}(x(t))+C_{\sigma(t)}\overline{f}(x(t-\tau(t)))\\ \qquad +J_{\sigma(t)},t>0,\\ y(t)=D_{\sigma(t)}x(t)+E_{\sigma(t)}x(t-\tau(t)),\\ x(t)=\overline{\varphi}(t),t\in[-\gamma,0],\end{cases}\tag{1}$$

where $x(t)=(x_1(t),x_2(t),\cdots,x_n(t))^T\in\mathbb{R}^n$ denotes the neural state vector of the system, $y(t)\in\mathbb{R}^q$ is the measured output, $\overline{f}(x(t))=(\overline{f}_1(x_1(t)),\overline{f}_2(x_2(t)),\cdots,\overline{f}_n(x_n(t)))^T\in\mathbb{R}^n$ denotes the neuron activation function and $J_{\sigma(t)}=(J_{1\sigma(t)},J_{2\sigma(t)},\cdots,J_{n\sigma(t)})^T\in\mathbb{R}^n$ is an external input. $A_{\sigma(t)}$, $B_{\sigma(t)}$, $C_{\sigma(t)}$, $D_{\sigma(t)}$ and $E_{\sigma(t)}$ are known real matrices with appropriate dimensions. Specifically, $A_{\sigma(t)}$ is a positive diagonal matrix, and $B_{\sigma(t)}\in\mathbb{R}^{n\times n}$, $C_{\sigma(t)}\in\mathbb{R}^{n\times n}$ are the connection weight matrices. The delay function $\tau(t)$ satisfies $0<\tau(t)\le\tau$, where τ is a positive constant. δ is a leakage delay and $\gamma=\tau\vee\delta$. $\overline{\varphi}(t)\in C^1([-\gamma,0],\mathbb{R}^n)$ is an initial function. $\sigma(t):[0,\infty)\to\mathbb{N}$ is a switching signal which is right continuous and piecewise constant function. As $\sigma(t)=i\in\mathbb{N}$, the ith subsystem is activated.

Throughout this thesis, we assume that the trajectory $x(t)$ is continuous everywhere. Additionally, the number of switching is finite in any interval.

Assumption 1. Every neural activation function satisfies

$$z_i^-\le\frac{\overline{f}_i(\beta_1)-\overline{f}_i(\beta_2)}{\beta_1-\beta_2}\le z_i^+,\ \forall\beta_1,\beta_2\in\mathbb{R},\ \beta_1\ne\beta_2,\ i\in\Lambda,$$

where z_i^- and z_i^+ are some real constants. For the convenience of demonstration, let

$$Z_1=\mathrm{diag}\{z_1^-z_1^+,z_2^-z_2^+,\cdots,z_n^-z_n^+\}$$

and

$$Z_2=\mathrm{diag}\{\frac{z_1^-+z_1^+}{2},\frac{z_2^-+z_2^+}{2},\cdots,\frac{z_n^-+z_n^+}{2}\}.$$

For SNNs (1), the following full-order state estimator is proposed as follows

$$\begin{cases}\dot{\hat{x}}(t)=-A_{\sigma(t)}\hat{x}(t-\delta)+B_{\sigma(t)}\overline{f}(\hat{x}(t))+C_{\sigma(t)}\overline{f}(\hat{x}(t-\tau(t)))\\ \qquad +J_{\sigma(t)}+L_{\sigma(t)}(y(t)-\hat{y}(t)),t>0,\\ \hat{y}(t)=D_{\sigma(t)}\hat{x}(t)+E_{\sigma(t)}\hat{x}(t-\tau(t)),\\ \hat{x}(t)=\overline{\psi}(t),t\in[-\gamma,0],\end{cases}\tag{2}$$

where $\hat{x}(t)\in\mathbb{R}^n$ is the state estimation of $x(t)$, $\overline{\psi}(t)\in C^1([-\gamma,0],\mathbb{R}^n)$ is an initial function, $L_{\sigma(t)}$ is the estimator gain matrix that needs to be designed. Let the estimation error be $\varpi(t)=x(t)-\hat{x}(t)$ and define functions as $\overline{f}_{\varpi}(t)=\overline{f}(x(t))-\overline{f}(\hat{x}(t))$, $\overline{f}_{\varpi}(t-\tau(t))=\overline{f}(x(t-\tau(t)))-\overline{f}(\hat{x}(t-\tau(t)))$. Hence, the estimation error system can be expressed in the following form by combining system (1) and (2)

$$\begin{cases}\dot{\varpi}(t) = -A_{\sigma(t)}\varpi(t-\delta) + B_{\sigma(t)}\overline{f}_{\varpi}(t) + C_{\sigma(t)}\overline{f}_{\varpi}(t-\tau(t)) \\ \qquad - L_{\sigma(t)}D_{\sigma(t)}\varpi(t) - L_{\sigma(t)}E_{\sigma(t)}\varpi(t-\tau(t)), t > 0, \\ \varpi(t) = \overline{\varphi}(t) - \overline{\psi}(t) = \overline{\eta}(t), t \in [-\gamma, 0]. \end{cases} \tag{3}$$

To prove the results, we need to draw into the below definitions and lemma.

Definition 1 ([27]). Given three positive constants $\overline{c}_1$, $\overline{c}_2$, T_0 with $\overline{c}_1 < \overline{c}_2$, a matrix $\overline{R} > 0$ and the switching signal $\sigma(t)$, estimation error system (3) is FTB with respect to $(\overline{c}_1, \overline{c}_2, T_0, \overline{R}, \sigma)$, if

$$\max\left(\sup_{-\gamma\le s\le 0} \overline{\eta}^T(s)\overline{R}\overline{\eta}(s),\ \sup_{-\gamma\le s\le 0} \dot{\overline{\eta}}^T(s)\overline{R}\dot{\overline{\eta}}(s)\right) \le \overline{c}_1$$

means that

$$\varpi^T(t)\overline{R}\varpi(t) < \overline{c}_2,\quad \forall t \in [0, T_0].$$

Definition 2 ([28]). For any $s \ge t \ge 0$ and the switching signal $\sigma(t)$, $N_\sigma(s,t)$ represents switching numbers in the interval $[t,s)$. If

$$N_\sigma(\mathrm{s},\mathrm{t}) \le \frac{s-t}{\tau_a} + N_1$$

holds for $\tau_a > 0$ and $N_1 \ge 0$, then the τ_a and N_1 is said to be average dwell time and chattering bound, respectively.

Lemma 1 ([29]). Given any matrix $Y > 0$ with appropriate dimensions and a function $\varsigma(\cdot):[a,b] \to \mathbb{R}^n$, such that the associated integrals are well defined, then

$$\left(\int_a^b \varsigma^T(s)ds\right) Y \left(\int_a^b \varsigma(s)ds\right) \leqslant (b-a)\int_a^b \varsigma^T(s) Y \varsigma(s)ds.$$

3. Main Results

In this section, by constructing suitable piecewise L-K functional and using ADT method, we now present the analysis results for the estimation error system (3) to be FTB.

Theorem 1. Under Assumption 1, the estimation error systems (3) is FTB with respect to $(\overline{c}_1, \overline{c}_2, T_0, \overline{R}, \sigma)$, if there exist positive scalars $\overline{c}_1$, $\overline{c}_2$, T_0 and α_i with $\overline{c}_1 < \overline{c}_2$, $n\times n$ matrices $\overline{P}_i > 0$, $\overline{Q}_i > 0$, $n\times n$ diagonal matrices $\overline{U}_{1i} > 0$, $\overline{U}_{2i} > 0$, $n\times n$ matrix $\overline{S}_i$ and $2n\times 2n$ matrix

$$\overline{T}_i = \begin{pmatrix} \overline{T}_{1i} & \overline{T}_{2i} \\ \star & \overline{T}_{3i} \end{pmatrix} > 0,$$

such that the following inequalities hold for any $i \in \mathbb{N}$,

$$\Pi_i = \begin{pmatrix} \Pi_{11}^i & \Pi_{12}^i & \Pi_{13}^i & \Pi_{14}^i & \Pi_{15}^i & \Pi_{16}^i \\ \star & \Pi_{22}^i & \Pi_{23}^i & \Pi_{24}^i & \Pi_{25}^i & \Pi_{26}^i \\ \star & \star & \Pi_{33}^i & 0 & 0 & \Pi_{36}^i \\ \star & \star & \star & \Pi_{44}^i & 0 & 0 \\ \star & \star & \star & \star & -\overline{U}_{1i} & 0 \\ \star & \star & \star & \star & \star & -\overline{U}_{2i} \end{pmatrix} < 0, \tag{4}$$

$$\overline{c}_1\kappa_2 \leqslant \overline{c}_2\kappa_1 e^{-\overline{\alpha}T_0}, \tag{5}$$

and the ADT τ_a satisfies

$$\tau_a > \tau_a^* = \frac{T_0 \ln h}{\ln(\overline{c}_2\kappa_1) - \overline{\alpha}T_0 - \ln(\overline{c}_1\kappa_2) - N_1 \ln h}, \tag{6}$$

where

$$\Pi_{11}^i = -\overline{P}_i L_i D_i - D_i^T L_i^T \overline{P}_i - \frac{1}{\delta}\overline{Q}_i - Z_1\overline{U}_{1i} - \alpha_i\overline{P}_i,$$

$$\Pi_{12}^i = -\overline{S}_i L_i D_i,\ \Pi_{13}^i = -\overline{P}_i L_i E_i + \overline{T}_{2i},$$

$$\Pi_{14}^i = -\overline{P}_i A_i + \frac{1}{\delta}\overline{Q}_i,\ \Pi_{15}^i = \overline{P}_i B_i + Z_2\overline{U}_{1i},\ \Pi_{16}^i = \overline{P}_i C_i,$$

$$\Pi_{22}^i = \tau\overline{T}_{3i} + \delta\overline{Q}_i - \overline{S}_i - \overline{S}_i^T,\ \Pi_{23}^i = -\overline{S}_i L_i E_i,$$

$$\Pi_{24}^i = -\overline{S}_i A_i,\ \Pi_{25}^i = \overline{S}_i B_i,\ \Pi_{26}^i = \overline{S}_i C_i,$$

$$\Pi_{33}^i = \tau\overline{T}_{1i} - \overline{T}_{2i} - \overline{T}_{2i}^T - Z_1\overline{U}_{2i},$$

$$\Pi_{36}^i = Z_2\overline{U}_{2i},\ \Pi_{44}^i = -\frac{1}{\delta}\overline{Q}_i,$$

$$\kappa_1 = \min_{i\in\mathbb{N}}(\lambda_{\min}(\overline{P}_i)),$$

$$\kappa_2 = \left(\max_{i\in\mathbb{N}}(\lambda_{\max}(\overline{P}_i)) + \tau^2 \max_{i\in\mathbb{N}}(\lambda_{\max}(\overline{T}_{3i})) + \delta^2 \max_{i\in\mathbb{N}}(\lambda_{\max}(\overline{Q}_i))\right) \cdot \frac{\lambda_{\max}(\overline{R})}{\lambda_{\min}(\overline{R})},$$

$$\overline{\alpha} = \max_{i\in\mathbb{N}}\{\alpha_i\},\ h = \max\{h_1, h_2, h_3, h_4\},$$

with

$$h_1 = \frac{\max_{i\in\mathbb{N}}(\lambda_{\max}(\overline{P}_i))}{\min_{i\in\mathbb{N}}(\lambda_{\min}(\overline{P}_i))},\ h_2 = \frac{\max_{i\in\mathbb{N}}(\lambda_{\max}(\overline{T}_i))}{\min_{i\in\mathbb{N}}(\lambda_{\min}(\overline{T}_i))},$$

$$h_3 = \frac{\max_{i\in\mathbb{N}}(\lambda_{\max}(\overline{T}_{3i}))}{\min_{i\in\mathbb{N}}(\lambda_{\min}(\overline{T}_{3i}))},\ h_4 = \frac{\max_{i\in\mathbb{N}}(\lambda_{\max}(\overline{Q}_i))}{\min_{i\in\mathbb{N}}(\lambda_{\min}(\overline{Q}_i))}.$$

Proof. Construct L-K functional

$$V_{\sigma(t)}(t)=\sum_{k=1}^{4}V_{k\sigma(t)}(t),$$

where

$$V_{1\sigma(t)}(t)=\varpi^T(t)\overline{P}_{\sigma(t)}\varpi(t),$$

$$V_{2\sigma(t)}(t)=\int_0^t\int_{\theta-\tau(\theta)}^{\theta}\begin{pmatrix}\varpi(\theta-\tau(\theta))\\ \dot{\varpi}(s)\end{pmatrix}^T\begin{pmatrix}\overline{T}_{1\sigma(t)} & \overline{T}_{2\sigma(t)}\\ \star & \overline{T}_{3\sigma(t)}\end{pmatrix}\begin{pmatrix}\varpi(\theta-\tau(\theta))\\ \dot{\varpi}(s)\end{pmatrix}dsd\theta,$$

$$V_{3\sigma(t)}(t)=\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\varpi}^T(s)\overline{T}_{3\sigma(t)}\dot{\varpi}(s)dsd\theta,$$

$$V_{4\sigma(t)}(t)=\int_{-\delta}^{0}\int_{t+\theta}^{t}\dot{\varpi}^T(s)\overline{Q}_{\sigma(t)}\dot{\varpi}(s)dsd\theta.$$

Assume that $\sigma(t)=i$, $t\in[t_k,t_{k+1})$ and $\sigma(t_k^-)=j$, $(i,j\in\mathbb{N})$. Along the trajectories of system (3), the derivative of V_{ki} $(k=1,2,3,4)$ can be obtained

$$\begin{aligned}\dot{V}_{1i}(t)&=-2\varpi^T(t)\overline{P}_iA_i\varpi(t-\delta)+2\varpi^T(t)\overline{P}_iB_i\overline{f}_{\varpi}(t)\\&+2\varpi^T(t)\overline{P}_iC_i\overline{f}_{\varpi}(t-\tau(t))\\&+\varpi^T(t)(-\overline{P}_iL_iD_i-D_i^TL_i^T\overline{P}_i)\varpi(t)\\&-2\varpi^T(t)\overline{P}_iL_iE_i\varpi(t-\tau(t)),\end{aligned}\tag{7}$$

$$\begin{aligned}\dot{V}_{2i}(t)&=\int_{t-\tau(t)}^{t}\begin{pmatrix}\varpi(t-\tau(t))\\ \dot{\varpi}(s)\end{pmatrix}^T\begin{pmatrix}\overline{T}_{1i} & \overline{T}_{2i}\\ \star & \overline{T}_{3i}\end{pmatrix}\begin{pmatrix}\varpi(t-\tau(t))\\ \dot{\varpi}(s)\end{pmatrix}ds\\&=\tau(t)\varpi^T(t-\tau(t))\overline{T}_{1i}\varpi(t-\tau(t))+2\varpi^T(t-\tau(t))\overline{T}_{2i}\varpi(t)\\&-2\varpi^T(t-\tau(t))\overline{T}_{2i}\varpi(t-\tau(t))+\int_{t-\tau(t)}^{t}\dot{\varpi}^T(s)\overline{T}_{3i}\dot{\varpi}(s)ds\\&\le\varpi^T(t-\tau(t))(\tau\overline{T}_{1i}-\overline{T}_{2i}-\overline{T}_{2i}^T)\varpi(t-\tau(t))\\&+2\varpi^T(t-\tau(t))\overline{T}_{2i}\varpi(t)+\int_{t-\tau}^{t}\dot{\varpi}^T(s)\overline{T}_{3i}\dot{\varpi}(s)ds,\end{aligned}\tag{8}$$

$$\begin{aligned}\dot{V}_{3i}(t)&=\int_{-\tau}^{0}(\dot{\varpi}^T(t)\overline{T}_{3i}\dot{\varpi}(t)-\dot{\varpi}(t+\theta)\overline{T}_{3i}\dot{\varpi}(t+\theta))d\theta\\&=\tau\dot{\varpi}^T(t)\overline{T}_{3i}\dot{\varpi}(t)-\int_{-\tau}^{0}\dot{\varpi}^T(t+\theta)\overline{T}_{3i}\dot{\varpi}(t+\theta)d\theta\\&=\tau\dot{\varpi}^T(t)\overline{T}_{3i}\dot{\varpi}(t)-\int_{t-\tau}^{t}\dot{\varpi}^T(s)\overline{T}_{3i}\dot{\varpi}(s)ds,\end{aligned}\tag{9}$$

$$\begin{aligned}\dot{V}_{4i}(t)&=\int_{-\delta}^{0}(\dot{\varpi}^T(t)\overline{Q}_i\dot{\varpi}(t)-\dot{\varpi}(t+\theta)\overline{Q}_i\dot{\varpi}(t+\theta))d\theta\\&=\delta\dot{\varpi}^T(t)\overline{Q}_i\dot{\varpi}(t)-\int_{-\delta}^{0}\dot{\varpi}^T(t+\theta)\overline{Q}_i\dot{\varpi}(t+\theta)d\theta\\&=\delta\dot{\varpi}^T(t)\overline{Q}_i\dot{\varpi}(t)-\int_{t-\delta}^{t}\dot{\varpi}^T(s)\overline{Q}_i\dot{\varpi}(s)ds.\end{aligned}$$

By Lemma 1, there is

$$\begin{aligned}-\int_{t-\delta}^{t}\dot{\varpi}^T(s)\overline{Q}_i\dot{\varpi}(s)ds&\le-\frac{1}{\delta}\left(\int_{t-\delta}^{t}\dot{\varpi}(s)ds\right)^T\overline{Q}_i\left(\int_{t-\delta}^{t}\dot{\varpi}(s)ds\right)\\&=-\frac{1}{\delta}(\varpi^T(t)-\varpi^T(t-\delta))\overline{Q}_i(\varpi(t)-\varpi(t-\delta))\\&=-\frac{1}{\delta}\varpi^T(t)\overline{Q}_i\varpi(t)+\frac{2}{\delta}\varpi^T(t)\overline{Q}_i\varpi(t-\delta)\\&-\frac{1}{\delta}\varpi^T(t-\delta)\overline{Q}_i\varpi(t-\delta),\end{aligned}$$

then

$$\begin{aligned}\dot{V}_{4i}(t)&\le\delta\dot{\varpi}^T(t)\overline{Q}_i\dot{\varpi}(t)-\frac{1}{\delta}\varpi^T(t)\overline{Q}_i\varpi(t)\\&+\frac{2}{\delta}\varpi^T(t)\overline{Q}_i\varpi(t-\delta)-\frac{1}{\delta}\varpi^T(t-\delta))\overline{Q}_i\varpi(t-\delta).\end{aligned}\tag{10}$$

In addition, note that

$$\begin{aligned}0&=2\dot{\varpi}^T(t)\overline{S}_i(-\dot{\varpi}(t)+\dot{\varpi}(t))\\&=\dot{\varpi}^T(t)(-\overline{S}_i-\overline{S}_i^T)\dot{\varpi}(t)-2\dot{\varpi}^T(t)\overline{S}_iA_i\varpi(t-\delta)\\&+2\dot{\varpi}^T(t)\overline{S}_iB_i\overline{f}_{\varpi}(t)+2\dot{\varpi}^T(t)\overline{S}_iC_i\overline{f}_{\varpi}(t-\tau(t))\\&-2\dot{\varpi}^T(t)\overline{S}_iL_iD_i\varpi(t)-2\dot{\varpi}^T(t)\overline{S}_iL_iE_i\varpi(t-\tau(t)).\end{aligned}\tag{11}$$

Moreover, for any $n\times n$ diagonal matrices $\overline{U}_{1i}>0$, $\overline{U}_{2i}>0$, We get from Assumption 1 that

$$\begin{pmatrix}\varpi(t)\\ \overline{f}_{\varpi}(t)\end{pmatrix}^T\begin{pmatrix}Z_1\overline{U}_{1i} & -Z_2\overline{U}_{1i}\\ \star & \overline{U}_{1i}\end{pmatrix}\begin{pmatrix}\varpi(t)\\ \overline{f}_{\varpi}(t)\end{pmatrix}\le0\tag{12}$$

and

$$\begin{pmatrix}\varpi(t-\tau(t))\\ \overline{f}_{\varpi}(t-\tau(t))\end{pmatrix}^T\begin{pmatrix}Z_1\overline{U}_{2i} & -Z_2\overline{U}_{2i}\\ \star & \overline{U}_{2i}\end{pmatrix}\begin{pmatrix}\varpi(t-\tau(t))\\ \overline{f}_{\varpi}(t-\tau(t))\end{pmatrix}\le0.\tag{13}$$

From (7)-(13), one has

$$\begin{aligned}\dot{V}_i(t)&\le\varpi^T(t)(-\overline{P}_iL_iD_i-D_i^TL_i^T\overline{P}_i-\frac{1}{\delta}\overline{Q}_i-Z_1\overline{U}_{1i})\varpi(t)\\&-2\dot{\varpi}^T(t)\overline{S}_iL_iD_i\varpi(t)\\&+2\varpi^T(t)(-\overline{P}_iL_iE_i+\overline{T}_{2i})\varpi(t-\tau(t))\\&+2\varpi^T(t)(-\overline{P}_iA_i+\frac{1}{\delta}\overline{Q}_i)\varpi(t-\delta)\\&+2\varpi^T(t)(\overline{P}_iB_i+Z_2\overline{U}_{1i})\overline{f}_{\varpi}(t)\\&+2\varpi^T(t)\overline{P}_iC_i\overline{f}_{\varpi}(t-\tau(t))\\&+\dot{\varpi}^T(t)(\tau\overline{T}_{3i}+\delta\overline{Q}_i-\overline{S}_i-\overline{S}_i^T)\dot{\varpi}(t)\\&-2\dot{\varpi}^T(t)\overline{S}_iL_iE_i\varpi(t-\tau(t))\\&-2\dot{\varpi}^T(t)\overline{S}_iA_i\varpi(t-\delta)\\&+2\dot{\varpi}^T(t)\overline{S}_iB_i\overline{f}_{\varpi}(t)\\&+2\dot{\varpi}^T(t)\overline{S}_iC_i\overline{f}_{\varpi}(t-\tau(t))\\&+\varpi^T(t-\tau(t))(\tau\overline{T}_{1i}-\overline{T}_{2i}-\overline{T}_{2i}^T-Z_1\overline{U}_{2i})\varpi(t-\tau(t))\end{aligned}$$

$$
\begin{aligned}
&+2\varpi^T(t-\tau(t))Z_2\bar{U}_{2i}\bar{f}_\varpi(t-\tau(t)) \\
&-\frac{1}{\delta}\varpi^T(t-\delta)\bar{Q}_i\varpi(t-\delta) \\
&-\bar{f}_\varpi^T(t)\bar{U}_{1i}\bar{f}_\varpi(t) \\
&-\bar{f}_\varpi^T(t-\tau(t))\bar{U}_{2i}\bar{f}_\varpi(t-\tau(t)) \\
=&\ \xi^T(t)\Pi_i\xi(t)+\alpha_iV_{1i}(t),
\end{aligned} \tag{14}
$$

where

$$
\xi(t)=\left(\varpi^T(t),\dot{\varpi}^T(t),\varpi^T(t-\tau(t)),\varpi^T(t-\delta),\bar{f}_\varpi^T(t),\bar{f}_\varpi^T(t-\tau(t))\right)^T,
$$

which, together with (4), yields that

$$
\dot{V}_i(t)\le\alpha_iV_{1i}(t)\le\alpha_iV_i(t). \tag{15}
$$

Integrating both sides of (15) from t_k to t, it can be obtained that

$$
V_i(t)\le e^{\alpha_i(t-t_k)}V_i(t_k)\le e^{\bar{\alpha}(t-t_k)}V_i(t_k),t\in[t_k,t_{k+1}). \tag{16}
$$

On the other hand, $\forall\varpi\in\mathbb{R}^n$, it holds that

$$
\varpi^T(t)\bar{P}_i\varpi(t)\le\lambda_{\max}(\bar{P}_i)\varpi^T(t)\varpi(t)\le\frac{\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{P}_i))}{\min_{i\in\mathbb{N}}(\lambda_{\min}(\bar{P}_i))}\varpi^T(t)\bar{P}_j\varpi(t),
$$

then, it yields

$$
V_{1i}\le h_1V_{1j}.
$$

Using similarity estimation, the following inequality can be obtained

$$
V_{2i}\le h_2V_{2j},\ V_{3i}\le h_3V_{3j},\ V_{4i}\le h_4V_{4j}.
$$

Denote $h=\max\{h_1,h_2,h_3,h_4\}$, one has

$$
V_i(t)\le hV_j(t),
$$

which implies that

$$
V_{\sigma(t_k)}(t_k)\le hV_{\sigma(t_k^-)}(t_k^-). \tag{17}
$$

Hence, for $h\ge1$, $t\in[0,T_0)$, substituting (17) into (16), we get from the definition of ADT that

$$
\begin{aligned}
V_{\sigma(t)}(t)&\le h^{N_\sigma(T_0,0)}V_{\sigma(0)}(0)e^{\bar{\alpha}(T_0-0)} \\
&\le e^{\bar{\alpha}T_0}h^{\frac{T_0}{\tau_a}+N_1}V_{\sigma(0)}(0).
\end{aligned} \tag{18}
$$

Furthermore, it is clear that

$$
\begin{aligned}
V_{\sigma(t)}(t)&\ge\lambda_{\min}(\bar{P}_i)\varpi^T(t)\varpi(t) \\
&\ge\min_{i\in\mathbb{N}}(\lambda_{\min}(\bar{P}_i))\varpi^T(t)\varpi(t) \\
&=\kappa_1\varpi^T(t)\varpi(t),
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
V_{\sigma(0)}(0)&=\varpi^T(0)\bar{P}_{\sigma(0)}\varpi(0)+\int_{-\tau}^0\int_\theta^0\dot{\varpi}^T(s)\bar{T}_{3\sigma(0)}\dot{\varpi}(s)dsd\theta \\
&\quad+\int_{-\delta}^0\int_\theta^0\dot{\varpi}^T(s)\bar{Q}_{\sigma(0)}\dot{\varpi}(s)dsd\theta \\
&\le\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{P}_i))\varpi^T(0)\varpi(0)+(\tau^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{T}_{3i})) \\
&\quad+\delta^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{Q}_i)))\cdot\sup_{-\gamma\le s\le0}\dot{\varpi}^T(s)\dot{\varpi}(s) \\
&\le(\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{P}_i)+\tau^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{T}_{3i})) \\
&\quad+\delta^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{Q}_i)))\cdot\frac{1}{\lambda_{\min}(\bar{R})} \\
&\quad\cdot\max\left(\sup_{-\gamma\le s\le0}\bar{\eta}^T(s)\bar{R}\bar{\eta}(s),\ \sup_{-\gamma\le s\le0}\dot{\eta}^T(s)\bar{R}\dot{\eta}(s)\right).
\end{aligned} \tag{20}
$$

By combining (18)-(20), what can be concluded is that

$$
\begin{aligned}
&\frac{\kappa_1}{\lambda_{\max}(\bar{R})}\varpi^T(t)\bar{R}\varpi(t)\le\kappa_1\varpi^T(t)\varpi(t)\le V_{\sigma(t)}(t) \\
&\le e^{\bar{\alpha}T_0}h^{\frac{T_0}{\tau_a}+N_1}V_{\sigma(0)}(0) \\
&\le e^{\bar{\alpha}T_0}h^{\frac{T_0}{\tau_a}+N_1}(\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{P}_i)) \\
&\quad+\tau^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{T}_{3i})) \\
&\quad+\delta^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{Q}_i)))\cdot\frac{1}{\lambda_{\min}(\bar{R})} \\
&\quad\cdot\max\left(\sup_{-\gamma\le s\le0}\bar{\eta}^T(s)\bar{R}\bar{\eta}(s),\ \sup_{-\gamma\le s\le0}\dot{\eta}^T(s)\bar{R}\dot{\eta}(s)\right).
\end{aligned} \tag{21}
$$

From Definition 1, it follows

$$
\begin{aligned}
\frac{\kappa_1}{\lambda_{\max}(\bar{R})}\varpi^T(t)\bar{R}\varpi(t)&\le e^{\bar{\alpha}T_0}h^{\frac{T_0}{\tau_a}+N_1}(\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{P}_i)) \\
&\quad+\tau^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{T}_{3i})) \\
&\quad+\delta^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{Q}_i)))\cdot\frac{1}{\lambda_{\min}(\bar{R})}\cdot\bar{c}_1.
\end{aligned} \tag{22}
$$

From (22), we can get

$$
\begin{aligned}
\varpi^T(t)\bar{R}\varpi(t)&\le e^{\bar{\alpha}T_0}h^{\frac{T_0}{\tau_a}+N_1}(\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{P}_i)) \\
&\quad+\tau^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{T}_{3i})) \\
&\quad+\delta^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{Q}_i))) \\
&\quad\cdot\frac{\lambda_{\max}(\bar{R})}{\lambda_{\min}(\bar{R})}\cdot\frac{1}{\kappa_1}\cdot\bar{c}_1 \\
&\le e^{\bar{\alpha}T_0}h^{\frac{T_0}{\tau_a}+N_1}\cdot\frac{\kappa_2}{\kappa_1}\cdot\bar{c}_1.
\end{aligned} \tag{23}
$$

When $h=1$, combining (5) and (23), one has

$$\varpi^T(t)\bar{R}\varpi(t) \le e^{\bar{\alpha}T_0}\frac{\kappa_2}{\kappa_1}\cdot\bar{c}_1 \le \bar{c}_2. \tag{24}$$

While $h>1$, combining (6) and (23), one has

$$\left(\frac{T_0}{\tau_\alpha}+N_1\right)\ln(h) \le \ln\left(\frac{\kappa_1\bar{c}_2}{\kappa_2\bar{c}_1}e^{-\bar{\alpha}T_0}\right),$$

which means that

$$\varpi^T(t)\bar{R}\varpi(t) \le \bar{c}_2. \tag{25}$$

Hence, for $h \ge 1$, according to (24) and (25), it follows

$$\varpi^T(t)\bar{R}\varpi(t) \le \bar{c}_2,\ \ t\in[0,T_0].$$

Therefore, the estimation error system (3) is FTB with respect to $(\bar{c}_1,\bar{c}_2,T_0,\bar{R},\sigma)$, which completes the proof.

Remark 1. Until now, a lot of interesting results concerning the state estimation of SNNs with time-varying delays have been reported [30-32]. Note that all the above results focus on the systems with differentiable time delays $\tau(t)$, which cannot be applied to the systems with uncharted or inestimable time delays. In our work, by using the appropriate L-K functional related to the derivative of the variable of state and leading into the auxiliary equation in (11), the differentiability restriction of the time delays imposed in the above literatures is eliminated completely.

It should be noted that Theorem 1 is obtained when the gain matrix of estimator is known. Actually, $L_{\sigma(t)}$ is difficult to know in advance and needs to be determined in this circumstance. Therefore, the method to design the state estimator gain is given in the following.

Theorem 2. Under Assumption 1, the estimation error system (3) is FTB with respect to $(\bar{c}_1,\bar{c}_2,T_0,\bar{R},\sigma)$, if there exist scalars $\bar{c}_1$, $\bar{c}_2$, T_0, α_i, μ_i with $\bar{c}_1<\bar{c}_2$, $n\times n$ matrices $\bar{P}_i>0$, $\bar{Q}_i>0$, $n\times n$ diagonal matrices $\bar{U}_{1i}>0$, $\bar{U}_{2i}>0$, $n\times n$ matrix $\bar{X}_i$ and $2n\times 2n$ matrix

$$\bar{T}_i=\begin{pmatrix}\bar{T}_{1i} & \bar{T}_{2i}\\ \star & \bar{T}_{3i}\end{pmatrix}>0,$$

such that the following inequalities hold for any $i\in\mathbb{N}$,

$$\tilde{\Pi}_i=\begin{pmatrix}\tilde{\Pi}^i_{11} & \tilde{\Pi}^i_{12} & \tilde{\Pi}^i_{13} & \Pi^i_{14} & \Pi^i_{15} & \Pi^i_{16}\\ \star & \tilde{\Pi}^i_{22} & \tilde{\Pi}^i_{23} & \tilde{\Pi}^i_{24} & \tilde{\Pi}^i_{25} & \tilde{\Pi}^i_{26}\\ \star & \star & \Pi^i_{33} & 0 & 0 & \Pi^i_{36}\\ \star & \star & \star & \Pi^i_{44} & 0 & 0\\ \star & \star & \star & \star & -\bar{U}_{1i} & 0\\ \star & \star & \star & \star & \star & -\bar{U}_{2i}\end{pmatrix}<0, \tag{26}$$

$$\bar{c}_1\kappa_2 \le \bar{c}_2\kappa_1 e^{-\bar{\alpha}T_0}, \tag{27}$$

and the ADT τ_a satisfies

$$\tau_a>\tau_a^*=\frac{T_0\ln h}{\ln(\bar{c}_2\kappa_1)-\bar{\alpha}T_0-\ln(\bar{c}_1\kappa_2)-N_1\ln h}, \tag{28}$$

where

$$\tilde{\Pi}^i_{11}=-\bar{X}_iD_i-D_i^T\bar{X}_i^T-\frac{1}{\delta}\bar{Q}_i-Z_1\bar{U}_{1i}-\alpha_i\bar{P}_i,$$

$$\tilde{\Pi}^i_{12}=-\mu_i\bar{X}_iD_i,\ \ \tilde{\Pi}^i_{13}=-\bar{X}_iE_i+\bar{T}_{2i},$$

$$\Pi^i_{14}=-\bar{P}_iA_i+\frac{1}{\delta}\bar{Q}_i,\ \ \Pi^i_{15}=\bar{P}_iB_i+Z_2\bar{U}_{1i},$$

$$\Pi^i_{16}=\bar{P}_iC_i,\ \ \tilde{\Pi}^i_{22}=\tau\bar{T}_{3i}+\delta\bar{Q}_i-2\mu_i\bar{P}_i,$$

$$\tilde{\Pi}^i_{23}=-\mu_i\bar{X}_iE_i,\ \ \tilde{\Pi}^i_{24}=-\mu_i\bar{P}_iA_i,\ \ \tilde{\Pi}^i_{25}=\mu_i\bar{P}_iB_i,$$

$$\tilde{\Pi}^i_{26}=\mu_i\bar{P}_iC_i,\ \ \Pi^i_{33}=\tau\bar{T}_{1i}-\bar{T}_{2i}-\bar{T}_{2i}^T-Z_1\bar{U}_{2i},$$

$$\Pi^i_{36}=Z_2\bar{U}_{2i},\ \ \Pi^i_{44}=-\frac{1}{\delta}\bar{Q}_i,$$

$$\kappa_1=\min_{i\in\mathbb{N}}(\lambda_{\min}(\bar{P}_i)),$$

$$\kappa_2=\left(\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{P}_i))+\tau^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{T}_{3i}))+\delta^2\max_{i\in\mathbb{N}}(\lambda_{\max}(\bar{Q}_i))\right)\cdot\frac{\lambda_{\max}(\bar{R})}{\lambda_{\min}(\bar{R})},$$

$$\bar{\alpha}=\max_{i\in\mathbb{N}}\{\alpha_i\},\ \ h=\max\{h_1,h_2,h_3,h_4\}.$$

Furthermore, the applicable state estimator gain is $L_i=\bar{P}_i^{-1}\bar{X}_i$.

Proof. By defining $\bar{S}_i=\mu_i\bar{P}_i$ and $\bar{P}_iL_i=\bar{X}_i$, inequalities (4) in Theorem 1 are reduced to (26). Then the state estimator gain can be designed as $L_i=\bar{P}_i^{-1}\bar{X}_i$. The proof is done.

Remark 2. The FTSE issue for various kind of delayed neural networks have been studied in recent achievements [20-22]. Compared with them, very little attention was paid to the corresponding researches for delayed SNNs. The authors studied the FTSE in H_∞ sense for SNNs with time-varying delays by making use of L-K functional coupled with ADT method [32]. However, leakage delay is excluded, and time delays are required to be differentiable in their results. In our work, the conditions established in Theorem 2 rely both on the upper bound of time-varying delays and leakage delay. The estimator gain can be obtained by solving a family of LMIs without requiring time-varying delays to be differentiable.

Especially, when $\delta=0$, the estimation error system (3) reduces to

$$\begin{cases}\dot{\varpi}(t)=-(A_{\sigma(t)}+L_{\sigma(t)}D_{\sigma(t)})\varpi(t)+B_{\sigma(t)}\bar{f}_\varpi(t)\\ \qquad +C_{\sigma(t)}\bar{f}_\varpi(t-\tau(t))-L_{\sigma(t)}E_{\sigma(t)}\varpi(t-\tau(t)),t>0,\\ \varpi(t)=\bar{\varphi}(t)-\bar{\psi}(t)=\bar{\eta}(t),t\in[-\gamma,0],\end{cases} \tag{29}$$

Then the following corollary can be obtained.

Corollary 1. Under Assumption 1, the estimation error system (3) is FTB with respect to $(\bar{c}_1, \bar{c}_2, T_0, \bar{R}, \sigma)$, if there exist scalars $\bar{c}_1$, $\bar{c}_2$, T_0, α_i, μ_i with $\bar{c}_1 < \bar{c}_2$, $n \times n$ matrices $\bar{P}_i > 0$, $n \times n$ diagonal matrices $\bar{U}_{1i} > 0$, $\bar{U}_{2i} > 0$, $n \times n$ matrix $\bar{X}_i$ and $2n \times 2n$ matrix

$$\bar{T}_i = \begin{pmatrix} \bar{T}_{1i} & \bar{T}_{2i} \\ \star & \bar{T}_{3i} \end{pmatrix} > 0,$$

such that the following inequalities hold for any $i \in \mathbb{N}$,

$$\bar{\Pi}_i = \begin{pmatrix} \bar{\Pi}_{11}^i & \bar{\Pi}_{12}^i & \Pi_{13}^i & \bar{\Pi}_{14}^i & \bar{\Pi}_{15}^i \\ \star & \bar{\Pi}_{22}^i & \Pi_{23}^i & \bar{\Pi}_{24}^i & \bar{\Pi}_{25}^i \\ \star & \star & \Pi_{33}^i & 0 & 0 \\ \star & \star & \star & -\bar{U}_{1i} & 0 \\ \star & \star & \star & \star & -\bar{U}_{2i} \end{pmatrix} < 0, \tag{30}$$

$$\bar{c}_1 \bar{\kappa}_2 \le \bar{c}_2 \kappa_1 e^{-\bar{\alpha} T_0}, \tag{31}$$

and the ADT τ_a satisfies

$$\tau_a > \bar{\tau}_a^* = \frac{T_0 \ln \bar{h}}{\ln(\bar{c}_2 \kappa_1) - \bar{\alpha} T_0 - \ln(c_1 \bar{\kappa}_2) - N_1 \ln \bar{h}}, \tag{32}$$

where

$$\bar{\Pi}_{11}^i = -\bar{P}_i A_i - A_i^T \bar{P}_i - \bar{X}_i D_i - D_i^T \bar{X}_i^T - Z_1 \bar{U}_{1i} - \alpha_i \bar{P}_i,$$

$$\bar{\Pi}_{12}^i = -\mu_i \bar{P}_i A_i - \mu_i \bar{X}_i D_i, \quad \tilde{\Pi}_{13}^i = -\bar{X}_i E_i + \bar{T}_{2i},$$

$$\bar{\Pi}_{14}^i = \bar{P}_i B_i + Z_2 \bar{U}_{1i}, \quad \bar{\Pi}_{15}^i = \bar{P}_i C_i, \quad \bar{\Pi}_{22}^i = \tau \bar{T}_{3i} - 2\mu_i \bar{P}_i,$$

$$\tilde{\Pi}_{23}^i = -\mu_i \bar{X}_i E_i, \quad \bar{\Pi}_{24}^i = \mu_i \bar{P}_i B_i, \quad \bar{\Pi}_{25}^i = \mu_i \bar{P}_i C_i,$$

$$\Pi_{33}^i = \tau \bar{T}_{1i} - \bar{T}_{2i} - \bar{T}_{2i}^T - Z_1 \bar{U}_{2i}, \quad \bar{\Pi}_{35}^i = Z_2 \bar{U}_{2i},$$

$$\bar{h} = \max\{h_1, h_2, h_3\},$$

$$\bar{\kappa}_2 = \left(\max_{i \in \mathbb{N}} (\lambda_{\max}(\bar{P}_i)) + \tau^2 \max_{i \in \mathbb{N}} (\lambda_{\max}(\bar{T}_{3i})) \right) \frac{\lambda_{\max}(\bar{R})}{\lambda_{\min}(\bar{R})},$$

and other parameters are the same as in Theorem 2.

4. Illustrative Example

A numerical example with its simulation results is supplied to prove the validity and applicability of the raised results.

Consider estimation error system (3) with two neurons characterized by the following parameters

$$A_1 = \begin{pmatrix} 0.24 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0.55 & -0.3 \\ 0.2 & 0.9 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -0.6 & -0.9 \\ -0.1 & -0.5 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0.98 & 0.21 \\ -0.19 & 0.65 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0.5 & -0.7 \\ -0.9 & -0.3 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} 0.9 & -0.5 \\ -0.9 & 0.5 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -1.2 & 0.9 \\ -0.1 & -1.4 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} -0.8 & 0.47 \\ 0.81 & -0.58 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1.35 & -0.8 \\ 0.21 & 1.25 \end{pmatrix},$$

$$\bar{f}(s) = \begin{pmatrix} \tanh(0.6s) \\ \tanh(0.8s) \end{pmatrix}, \quad \tau(t) = 0.09 + 0.01 \sin t.$$

It can be seen that $Z_1 = \mathrm{diag}\{0, 0\}$, $Z_2 = \mathrm{diag}\{0.3, 0.4\}$. Given scalars $\tau = 0.1$, $\delta = 0.1$, $\mu_1 = 1.6$ and $\mu_2 = 1.6$. We consider the case that $\bar{c}_1 = 1$, $\bar{c}_2 = 30$, $T_0 = 10$, $\alpha_1 = 0.01$, $\alpha_2 = 0.01$ and $\bar{R} = I$. In order to solve the unknown matrices conveniently, the following positive definite diagonal matrices are given

$$\bar{U}_{11} = \begin{pmatrix} 1.5945 & 0 \\ 0 & 1.5945 \end{pmatrix}, \quad \bar{U}_{12} = \begin{pmatrix} 0.2571 & 0 \\ 0 & 0.2571 \end{pmatrix},$$

$$\bar{U}_{21} = \begin{pmatrix} 0.1409 & 0 \\ 0 & 0.1409 \end{pmatrix}, \bar{U}_{22} = \begin{pmatrix} 0.0105 & 0 \\ 0 & 0.0105 \end{pmatrix}.$$

Using the MATLAB LMI Toolbox to solve the LMIs in Theorem 2, we come up with the following feasible solutions:

$$\bar{P}_1 = \begin{pmatrix} 0.1047 & -0.0232 \\ -0.0232 & 0.1295 \end{pmatrix}, \quad \bar{P}_2 = \begin{pmatrix} 0.0784 & 0.0105 \\ 0.0105 & 0.0918 \end{pmatrix},$$

$$\bar{Q}_1 = \begin{pmatrix} 0.7766 & 0.0402 \\ 0.0402 & 0.9611 \end{pmatrix}, \quad \bar{Q}_2 = \begin{pmatrix} 0.7223 & 0.0146 \\ 0.0146 & 0.7256 \end{pmatrix},$$

$$\bar{X}_1 = \begin{pmatrix} 0.3556 & 0.2265 \\ -1.3359 & -1.2525 \end{pmatrix}, \quad \bar{X}_2 = \begin{pmatrix} -0.0192 & -0.0107 \\ 0.0126 & -0.0397 \end{pmatrix},$$

$$\bar{T}_1 = \begin{pmatrix} 0.8007 & -0.1441 & 0.2366 & -0.0284 \\ -0.1441 & 1.1250 & -0.0758 & 0.1332 \\ 0.2366 & -0.0758 & 0.8425 & 0.0074 \\ -0.0284 & 0.1332 & 0.0074 & 1.0588 \end{pmatrix},$$

$$\bar{T}_2 = \begin{pmatrix} 1.7493 & 0.1149 & 0.2778 & 0.0063 \\ 0.1149 & 1.7159 & 0.0114 & 0.3065 \\ 0.2778 & 0.0114 & 0.7250 & 0.0287 \\ 0.0063 & 0.3065 & 0.0287 & 0.7525 \end{pmatrix},$$

and the gain matrices of the state estimator are designed as

$$L_1 = \bar{P}_1^{-1} \bar{X}_1 = \begin{pmatrix} 1.1591 & 0.0244 \\ -10.1121 & -9.6703 \end{pmatrix},$$

$$L_2 = \overline{P}_2^{-1}\overline{X}_2 = \begin{pmatrix} -0.2670 & -0.0800 \\ 0.1674 & -0.4235 \end{pmatrix}.$$

In addition, the lower bound of ADT is calculated as $\tau_a^* = 6.3799$ and we choose $\tau_a = 6.4799$. By Theorem 2, the estimation error system (3) is FTB with respect to $(1, 30, 10, I, \sigma)$. When the initial functions are given by

$$\overline{\varphi}(t) = \begin{pmatrix} 0.12 \\ 0.15 \end{pmatrix}, \quad \overline{\psi}(t) = \begin{pmatrix} -0.55 \\ -0.54 \end{pmatrix},$$

the simulation results are shown in figures 1-4. Figures 1 and 2 are the real states $x_1(t)$ and $x_2(t)$ with their estimations $\hat{x}_1(t)$ and $\hat{x}_2(t)$, respectively. Figure 3 represents the trajectory of $\varpi^T(t)\overline{R}\varpi(t)$ and what can be seen is that the validity of Theorem 2 guarantees $\varpi^T(t)\overline{R}\varpi(t)$ below the given $\overline{c}_2$ in the interval $[0, T_0]$. Figure 4 depicts the switching signal $\sigma(t)$ and switching instants. The simulation results indicate that the state estimator can track the state of delayed SNNs, which verifies the validity and correctness of the developed method.

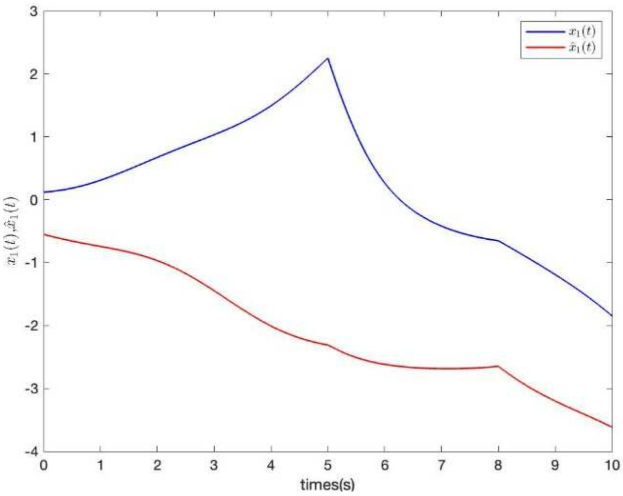

Figure 1. *The state trajectory of* $x_1(t)$ *and its estimation* $\hat{x}_1(t)$.

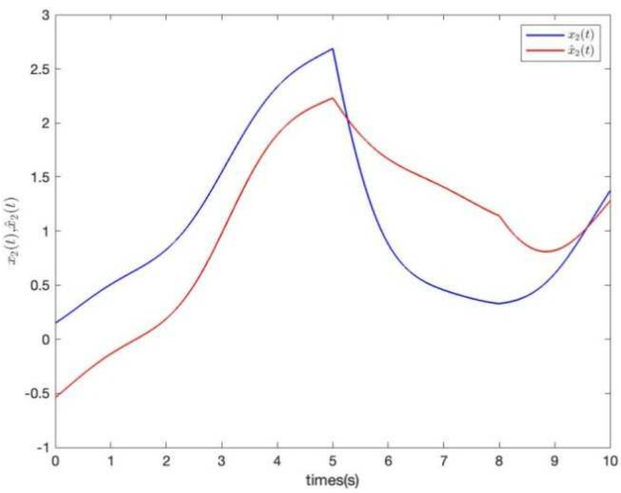

Figure 2. *The state trajectory of* $x_2(t)$ *and its estimation* $\hat{x}_2(t)$.

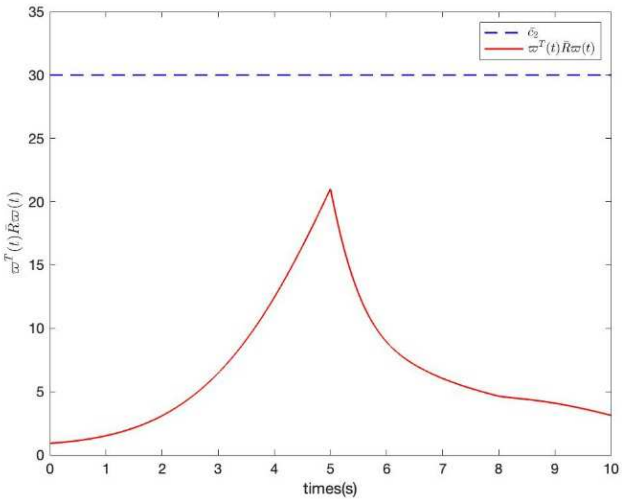

Figure 3. *The trajectory* $\varpi^T(t)\overline{R}\varpi(t)$ *of estimation error system (3).*

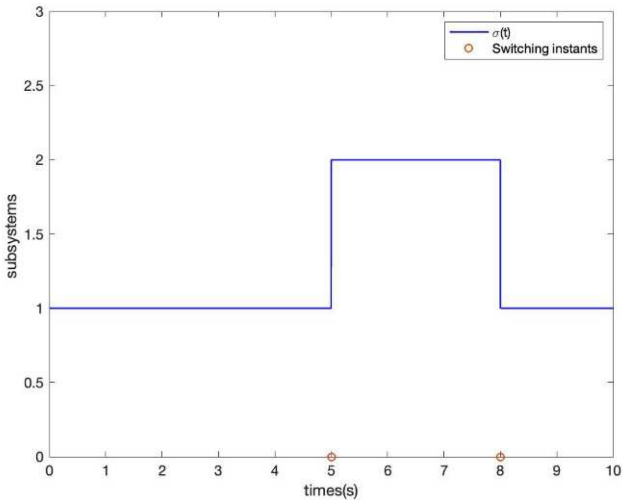

Figure 4. *The switching signal* $\sigma(t)$ *and switching instants.*

5. Conclusion

This article has gone into the issue of FTSE for a set of SNNs with leakage delay and time-varying delays. The L-K functional method and ADT switching laws have been employed to deduce some sufficient criteria, which can make the estimation error system (3) be FTB. The characterization of the estimator gain has been realized by solving certain LMIs. The obtained sufficient conditions are delay-dependent and do not require the time delays to be differentiable. Finally, an example is performed with its numerical simulations to corroborate the efficiency of the developed results. As we all know, SNNs have been studied extensively in various fields including biology, sociology and physics. Therefore, searching other analysis technique and other control methods for SNNs to acquire the less conservative results in finite-time sense will be our next work.

Acknowledgements

This work is supported by National Natural Science Foundation of China (11501333).

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