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Research Article

A Mathematical Model of Lassa Fever Transmission and Control in Ebonyi State, Nigeria

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Abstract

Lassa virus is transmitted from rodents to humans, but it is not known whether humans can transmit Lassa fever to rats. The virus is thought to spread to humans through contact with contaminated food or surfaces. Other forms of infection include handling rodents for food (people often get rodent blood and urine on their hands) and bites. It can also spread through the use of contaminated medical equipment, such as reusing needles. The state variables of the Lassa Fever model equation is expressed as nonlinear ordinary differential equations in the technique of an initial value problem (IVP) having 10 parameters. As a result of measuring the spread of Lassa fever and determining the stability equilibrium, Lassa fever was found to be stable at an equilibrium point ε_0 for which the basic reproduction number $R_0 < 1$. This paper optimized three control measures as a means to limit the spread of Lassa fever. The first two measures - regular hand washing and keeping homes and environment clean reduced the rate and impact of transmission between rodents and humans and the treatment of Lassa fever patients reduce transmission to human hosts, which were achieved by the operation of Pontryagin's Maximum Principle. Therefore, the results of this study demonstrate that the joint control measures adopted in this paper are effective strategies to combat the spread of disease.

Keywords

Lassa Fever, Scaling, Basic Reproduction Number, Stability Analysis, Controls

1. Introduction

Lassa fever which is an acute viral hemorrhagic disease belonging to several countries in West Africa, such as Benin, Ghana, Guinea, Liberia, Mali, Sierra Leone, and Nigeria [18]. The first case of this disease was reported in the 1950s, and the virus was identified in 1969 following the deaths of two missionary nurses in Lhasa, Nigeria. Lassa fever is endemic in Nigeria and outbreaks occur almost every year in different parts of the country [1, 3]. The Nigeria Center for Disease Control initiates regular report on Lassa fever supervision [2]. Lassa virus is present in wild rats that have multiple mammaries (udders) and excrete the virus in their urine and feces, They are common in rural areas of tropical Africa and often live in and around homes [5, 6]. The latent period for Lassa fever is 2-days to 3-weeks [5]. This infection is transmitted from rodents to humans and, to a lesser extent, from humans to humans. Infected rodents spread the virus throughout their lives and can spread the virus through urine, saliva, respiratory tract, and open blood vessels, even if they do not show clinical symptoms [4, 5]. Transmission from rodents to humans occurs through direct contact with the urine, feces, or saliva of infected rodents and discharge or secretion resulting from contact with infectious substances or consumption of

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food contaminated with feces [8]. In 2017, Innocent et al., formulated a Lassa fever model with measures to curtail it, studied the epidemiology of the disease, recommended avoidance of contact with virus-carrying rodents, and introducing human vaccines [12]. Martins et al., in their study developed a mathematical model to control the spread of lassa fever, analyzing the existence and stability of a lassa fever-free equilibrium [10]. Abdulkarim et al., discussed the objective factors and mortality rates lassa fevr in Bauchi State, Nigeria were data collected from 2015 to 2018 were used to show an increase in morbidity and mortality and the majority of deaths was shown to occurring within 7 days of symptom onset [11].

Patrick et al. [7] proposed a mathematical model of Lassa fever transmission dynamics that included isolation and treatment as control strategies, their numerical simulations showed that the rate of spread of infection is an important parameter for the emergence of infectious disease. Therefore, efforts should be made to minimize transmission parameters to ensure eradication. Another model for the isolation of symptomatic Lassa fever was presented in [9]. This study extends studies [7, 9] by optimizing the prescribed control measures adopted by MEDECINS SANS FRONTERES [15] in the areas of regular hand washing and food hygiene, keeping homes and community environments clean, and treating patients infected with Lassa fever. This article experiments with the Forward-Backward Sweep approach, which uses the order four Runge-Kutta method to confirm the effectiveness of the control measures by the operation of the Pontryagin's Maximum Principle to determine how the proliferate of the Lassa Fever can be limited. Additional, this article will also examines the incidence and recurrence rates of infection in Lassa fever survivors.

2. Assumptions of the Lassa Fever Model

a. All state variables and parameters are assumed to be positive.
b. The entire population is vulnerable to Lassa fever
c. There is believed to be an even mix infected and susceptible individuals.
d. Some recovered individuals may return to the susceptible class.
e. The entire population is at risk equal of Lassa Fever, regardless age or health condition.

2.1. Lassa Fever Model Equations

The state variables of the Lassa Fever model equation is expressed as nonlinear ordinary differential equations in the technique of an initial value problem (IVP) having 10 parameters. The human population is divided into four classes; Susceptible (S_H), Exposed (E_H), Infected (I_H), and Recovered (R_H) compartments and the rodent populations are divided into two; Susceptible (S_R), and Infected (I_R) at t respectively. The susceptible people S_H go to the exposed section E_H to update population of exposed class to $\frac{S_H}{N_H}(\alpha_1 I_H + \alpha_2 I_R)$. From the exposed population, $\alpha_3 E_H$ persons are transfer from E_H compartment to the infection ward I_H and, as a result of compliance with treatment and prevention measures, $\alpha_4 I_H$ persons move to the recovery group. Finally, the susceptible rodents S_R move to the infected compartment I_R and update the number of infected rodents to $\frac{\alpha_2 S_R I_R}{N_R}$ at t.

$$\left.\begin{aligned}
\frac{dS_H}{dt} &= \Lambda_H N_H - \frac{S_H}{N_H}(\alpha_1 I_H + \alpha_2 I_R) - \mu_H S_H + \alpha_5 R_H \\
\frac{dE_H}{dt} &= \frac{S_H}{N_H}(\alpha_1 I_H + \alpha_2 I_R) - (\alpha_3 + \mu_H) E_H \\
\frac{dI_H}{dt} &= \alpha_3 E_H - (\alpha_4 + \delta + \mu_H) I_H \\
\frac{dR_H}{dt} &= \alpha_4 I_H - (\alpha_5 + \mu_H) R_H \\
\frac{dS_R}{dt} &= \Lambda_R N_R - \frac{\alpha_2 S_R I_R}{N_R} - \mu_R S_R \\
\frac{dI_R}{dt} &= \frac{\alpha_2 S_R I_R}{N_R} - \mu_R I_R
\end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned}
N_H(t) &= S_H(t) + E_H(t) + I_H(t) + R_H(t) \\
N_R(t) &= S_R(t) + I_R(t)
\end{aligned}\right\} \quad (2)$$

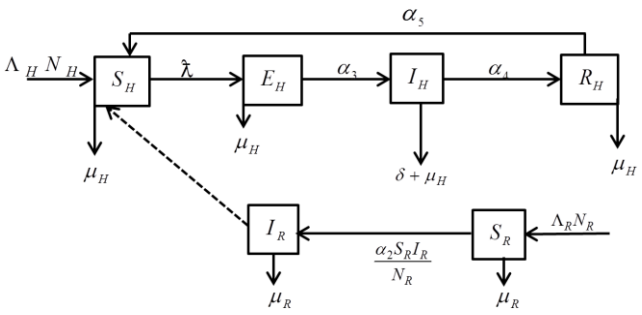

Figure 1. *Model Diagram.*

Where $\hat{\lambda} = \frac{S_H}{N_R}(\alpha_1 I_H + \alpha_2 I_R)$.

Table 1. *Summary of Parameters and meaning.*

Parameters	Meaning (Dimension: Time^{-1})
Λ_H	Recruitment rate for humans

Parameters	Meaning (Dimension: Time^{-1})
Λ_R	Recruitment rate for rodent
α_1	Contact rate of humans
α_2	Contact rate of rodents
α_3	Progression rate to the infectious class
α_4	Immunity lost rate
α_5	Rate at which recovered individuals go back to the susceptible class
δ	Lassa Fever induced death rate
μ_H	Natural death rate of human
μ_R	Natural death rate of rodents

2.2. Scaling of the Model

To facilitate the analysis of equation 1 of the Lassa fever model, the ratio of the give populations is determined by scaling the population of each class based on the total number of species. Considering different populations, $N(t)$, and the ratio of every section within the species is given as:

$$s_h = \frac{S_H}{N_H},\ e_h = \frac{E_H}{N_H},\ r_h = \frac{R_H}{N_H},\ s_r = \frac{S_R}{N_R}\ i_h = \frac{I_H}{N_H},$$

and $i_r = \frac{I_R}{N_R}$ it follows that $S_H = s_h N_H$, $E_H = e_h N_H$, $I_H = i_h N_H$, $R_H = r_h N_H$, $S_R = s_r N_R$, and $I_R = i_r N_R$.

Rat population was estimated 7 billion in the world and this means that 1 rat for every human [16] i.e

$\Lambda_h \geq \Lambda_r \Rightarrow \frac{\Lambda_h}{\Lambda_r} \geq 1$. Let $T_h = \frac{1}{\Lambda_h}$ and $T_r = \frac{1}{\Lambda_r}$ then $T_h \leq T_r$. Set $t = T_h t_h^*$ then $dt = T_h dt_h^*$.

$$N_H \frac{ds_h}{T_h dt_h^*} = \Lambda_H N_H - \frac{s_h}{N_H}\left(\alpha_1 i_h N_H + \alpha_2 i_r N_R\right) - \mu_H s_h N_H + \alpha_5 r_h N_H$$

$$N_H \frac{de_h}{T_h dt_h^*} = \frac{s_h}{N_H}\left(\alpha_1 i_h N_H + \alpha_2 i_r N_R\right) - \left(\alpha_3 + \mu_H\right) e_h N_H$$

$$N_H \frac{di_h}{T_h dt_h^*} = \alpha_3 e_h N_H - \left(\alpha_4 + \delta + \mu_H\right) i_h N_H$$

$$N_H \frac{dr_h}{T_h dt_h^*} = \alpha_4 i_h N_{HH} - \left(\alpha_5 + \mu_H\right) r_h N_H$$

$$N_R \frac{ds_r}{T_r dt_h^*} = \Lambda_R N_R - \frac{\alpha_2 s_r N_R i_r N_R}{N_R} - \mu_R s_r N_R$$

$$N_R \frac{dI_R}{T_r dt_h^*} = \frac{\alpha_2 s_r N_R i_r N_R}{N_R} - \mu_R i_r N_R$$

Then

$$\frac{ds_h}{T_h dt_h^*} = \Lambda_H - s_h\left(\alpha_1 i_h N_H + \alpha_2 i_r N_R\right) - \mu_H s_h + \alpha_5 r_h$$

$$\frac{de_h}{T_h dt_h^*} = s_h\left(\alpha_1 i_h N_H + \alpha_2 i_r N_R\right) - \left(\alpha_3 + \mu_H\right) e_h$$

$$\frac{di_h}{T_h dt_h^*} = \alpha_3 e_h - \left(\alpha_4 + \delta + \mu_H\right) i_h$$

$$\frac{dr_h}{T_h dt_h^*} = \alpha_4 i_h - \left(\alpha_5 + \mu_H\right) r_h$$

$$\frac{ds_r}{T_r dt_h^*} = \Lambda_R - \alpha_2 s_r i_r - \mu_R s_r$$

$$\frac{di_r}{T_r dt_h^*} = \alpha_2 s_r i_r - \mu_R i_r$$

After some simplification, we have

$$\left.\begin{aligned}
\frac{ds_h}{dt_h^*} &= 1 - \beta s_h i_h - \alpha s_h i_r - m s_h + \ell r_h \\
\frac{de_h}{dt_h^*} &= \beta s_h i_h + \alpha s_h i_r - \theta e_h - m e_h \\
\frac{di_h}{dt_h^*} &= \theta e_h - \kappa i_h - \varphi i_h - m i_h \\
\frac{dr_h}{dt_h^*} &= \kappa i_h - \ell r_h - m r_h \\
\frac{ds_r}{dt_h^*} &= 1 - \alpha s_r i_r - n s_r \\
\frac{di_r}{dt_h^*} &= \alpha s_r i_r - n i_r
\end{aligned}\right\} \quad (3)$$

Where the dimensionless parameters are $\beta = \frac{\alpha_1}{\Lambda_h}$, $\alpha = \frac{\alpha_2}{\Lambda_h}$, $m = \frac{\mu_h}{\Lambda_h}$, $\ell = \frac{\alpha_5}{\Lambda_h}$, $\theta = \frac{\alpha_3}{\Lambda_H}$, $\kappa = \frac{\alpha_4}{\Lambda_h}$, $\varphi = \frac{\delta}{\Lambda_h}$, $n = \frac{\mu_r}{\Lambda_h}$

2.3. Lassa Fever Model Properties

The Lassa Fever model (3) covers both the population of

humans and rodents, the variables and parameters of the Lassa fever model are all non-negative for $t \geq 0$.

Theorem 1: The Lassa fever model 1 of the initial condition in R_+^4 and R_+^2 are positively invariant in

$$\Omega_h = \left\{ (s_h, e_h, i_h, r_h) : s_h + e_h + i_h + r_h \leq \frac{1}{m} \right\} \text{ and}$$

$$\Omega_r = \left\{ (s_r, i_r) : s_r + i_r \leq \frac{1}{n} \right\}$$

Proof

The Lassa fever system (3) is split into two sections, i.e the class of humans N_H and the class of rodents N_R, defined by

$$N_h = s_h + e_h + i_h + r_h \tag{4}$$

$$N_r = s_r + i_r \tag{5}$$

Let $\Omega = \Omega_h \cup \Omega_r \in R_+^4 \times R_+^2 = R_+^6$

Then, equation (4), yields

$$\frac{dN_h}{dt} = 1 - mN_h - \varphi i_h$$

$$\frac{dN_h}{dt} \leq 1 - mN_h$$

$$\frac{dN_h}{dt} + mN_1 \leq 1$$

By method of integrating factor

$$I.F = e^{\int mdt} = e^{mt}$$

It follows as;

$$e^{mt} N_h(t) \leq \int e^{mt} + C$$

$$N_h(t) \leq \frac{1}{m} + Ce^{-mt}$$

$$\text{At } t = 0; \text{ then } C = N_h(0) - \frac{1}{m}$$

$$N_h(t) \leq \frac{\Lambda_h}{\mu} + e^{-\mu_H t}\left(N_h(0) - \frac{1}{m} \right)$$

When $t \to \infty$

$$N_h(t) \leq \frac{1}{m} \tag{6}$$

And equation (5), yield

$$\frac{dN_R}{dt} = 1 - ns_r - ni_r$$

$$\frac{dN_R}{dt} \leq 1 - nN_R$$

Applying method of integrating factor

$$I.F = e^{\int ndt} = e^{nt}$$

$$e^{nt} N_r(t) \leq \int e^{nt} + C_2$$

$$N_r \leq \frac{1}{n} + C_2 e^{-nt}$$

$$\text{At } t = 0,\ C_2 = N_r(0) - \frac{1}{n}$$

$$N_r(t) = \frac{1}{n} + e^{-nt}\left(N_r(0) - \frac{1}{n} \right)$$

When $t \to \infty$

$$N_r(t) \leq \frac{1}{n} \tag{7}$$

Therefore $N_h(t) \leq \frac{1}{m}$ and $N_r(t) \leq \frac{1}{n}$, then $N_h(0) \leq \frac{1}{m}$ and $N_r(0) \leq \frac{1}{n}$ respectively. This is an indication that the solutions of the Lassa Fever model (3) fall in the zone.

$$\Omega = \left\{ (s_h, e_h, i_h, r_h, s_r, i_r) \in R_+^6 : N_h \leq \frac{1}{m} \text{ and } N_r \leq \frac{1}{n} \right\}.$$

3. Existence of Lassa Fever Free Equilibrium and the Basic Reproduction Number

From the Lassa-Fever model equation 3, for the human population, the compartments s_h and r_h represent the disease-free states and e_h and i_h denote the infection class. The

Lassa Fever-free equilibrium (LF-FE) point $\varepsilon_0 = (s_h, e_h, i_h, r_h, s_r, i_r)$ by first setting $e_h = i_h = r_h = 0$ and $\frac{ds_h}{dt} = 1 - \beta s_h i_h - \alpha s_h i_r - m s_h + \ell r_h = 0.$

It follows that $s_h = \frac{1}{m}$ and for the rodents population, the compartments s_r is only the disease-free states and the compartments i_r is the infection class, $i_r = 0$ then set $\frac{ds_r}{dt} = 1 - \alpha s_r i_r - n s_r = 0$, $s_r = \frac{1}{n}$. Therefore, the Lassa Fever-free equilibrium (DFE) is $\varepsilon_0 = \left(\frac{1}{m}, 0, 0, 0, \frac{1}{n}, 0\right)$.

To obtain the basic reproduction number, R_0 of the model equation (3) at ε_0, the application of the next-generation matrix is employed [13]. As the infected compartments are e_h, i_h and i_r then F and V formed the ongoing infection terms and the out sending terms shown below.

$$F = \begin{bmatrix} 0 & \frac{\beta}{m} & \frac{\alpha}{n} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{n^2} \end{bmatrix} \text{ and } V = \begin{bmatrix} n(\kappa+\varphi+m) & n\theta & 0 \\ 0 & n(\theta+m) & 0 \\ 0 & 0 & (\theta+m)(\kappa+\varphi+m) \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} \frac{\beta\theta}{m(\theta+m)(\kappa+\varphi+m)} & \frac{\beta}{m(\kappa+\varphi+m)} & \frac{\alpha}{nm} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha}{n^2} \end{bmatrix}$$

It follows that the measure of transmission potential of Lassa Fever, denoted by R_0, is obtained from the matrix FV^{-1} by calculating it's spectral radius.

$$R_0 = \frac{\beta\theta}{m(\theta+m)(\kappa+\varphi+m)}$$

Theorem 2. When $R_0 < 1$, the Lassa Fever-free equilibrium ε_0 of the dynamical Lassa Fever equation 3 is locally asymptotically stable.

Proof

Simplifying the Jacobian matrix of the Lassa Fever equations at the Lassa Fever -free equilibrium point, the result is given by

$$J_{\varepsilon_0} = \begin{vmatrix} -m-\lambda & 0 & -\beta/m & \ell & 0 & -\alpha/m \\ 0 & -(\theta+m)-\lambda & \beta/m & 0 & 0 & \alpha/m \\ 0 & \theta & -(\kappa+\varphi+m)-\lambda & 0 & 0 & 0 \\ 0 & 0 & k & -(\ell+m)-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -n-\lambda & -\alpha/n \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{\alpha}{n}-n\right)-\lambda \end{vmatrix} = 0$$

$$\left[\left(\left(n-\frac{\alpha}{n}\right)+\lambda\right)\right][-n-\lambda]\left[(\ell+m)+\lambda\right][-m-\lambda]\left\{\left[-(\theta+m)-\lambda\right]\left[-(\kappa+\theta+m)-\lambda\right]-\frac{\beta\theta}{m}\right\} = 0$$

It follows that

$$\lambda_1 = \frac{\alpha}{n} - n$$

$$\lambda_2 = -n$$

$$\lambda_3 = -(\ell+m)$$

$$\lambda_4 = -m$$

$$\lambda_5 = -\frac{1}{2}\left(A + \sqrt{A^2+4B}\right)$$

$$\lambda_6 = -\frac{1}{2}\left(A - \sqrt{A^2+4B}\right)$$

Where

$A=2(\theta+m)+\kappa>0$ and $B=(\theta+m)(\kappa+\theta+m)-\frac{\beta\theta}{m}$

If $R_0<1$, then $B<0$ implies $\lambda_6<0$. All eigenvalues will zeros or negatives, hence ε_0 is locally stable. If $B>0$ implies $\lambda_6>0$ and ε_0 is locally unstable.

3.1. Global Stability of Lassa Fever - Free Equilibrium

The conditions for the global stability of the Lassa Fever model at ε_0 is obtained by applying the approach stated in [23, 24] which defines the human population N_h of the Lassa Fever model is defined by

$$\frac{dQ}{dt}=F(Q,S)$$
$$\frac{dS}{dt}=G(Q,S), G(Q,0)=0$$

Where $Q\in R^n$ denotes the class of individuals free from Lassa Fever and $S\in R^m$ denotes the infected individuals. From the above notation, the Lassa Fever-free equilibrium is written as $G_0=(Q,0)$. Then, these two conditions below clearly showed that the global stability of the Lassa Fever free equilibrium.

$p_1: \frac{dQ}{dt}F(Q,0)$, Q_0 is globally asymptotically stable

$p_2: G(Q,S)=BS-\hat{G}(Q,S)$, where $\hat{G}(Q,S)\geq 0$ for $Q,S\in\Omega$

Lemma 1: The equilibrium point $G_0=(Q,0)$ is globally asymptotically stable when $R_0\leq 1$ and the above assumptions on P_1 and P_2 are true.

Theorem 3: The Lassa Fever-free equilibrium point ε_0 of the Lassa Fever is globally asymptotically stable provided $R_0\leq 1$.

Proof

$$F(Q,0)=\begin{bmatrix}1-ms+\ell r_h\\ -mr_h\end{bmatrix}$$

$$\left|J_{F(Q,0)}-I\lambda\right|=\begin{bmatrix}-m & \ell\\ 0 & -m\end{bmatrix}$$

Solving for the characteristic polynomial of $J_{F(X,0)}$; we have

$$(-\lambda-m)(-\lambda-m)=0$$

$$\lambda_1=\lambda_2=-m$$

Therefore, $Q=Q_0$ is globally asymptotically stable.

Then,

$$G(Q,S)=BS-\hat{G}(Q,S),$$

We have

$$G(Q,S)=\begin{bmatrix}-(\theta+m) & \beta s_h\\ \theta & -(\kappa+\theta+m)\end{bmatrix}\begin{bmatrix}e_h\\ i_h\end{bmatrix}$$

Therefore, it follows that B satisfies all conditions stated in P_2.

3.2. Strategy for Prevention of Lassa Fever

The Preventive measures as adopted by MEDECINS SANS FRONTIERES, DOCTORS WITHOUT BORDERS [15] to curtail the spread of Lassa fever will be categorized as follows:

ξ Regular Hand Washing with soap and clean water and food Hygiene; wash vegetables and fruits before eating, properly cover food, Food should be properly cooked, store food in containers with lids or covers, kitchen utensils should be kept clean and covered, avoid hunting and eating of rats will be set to reduce the spread of Lassa fever in the human population.

η Maintaining a clean Environment at home and the community: keep a cat around, close holes in the house, use of door and window will be set to reduce the breeding of rats in the environment.

ω: Treatment of individuals infected with Lassa Fever.

3.3. Lassa Fever Model Equations with Controls

$$\left.\begin{aligned}
\frac{ds_h}{dt_h^*} &= 1-(1-\xi)\beta s_h i_h - \alpha s_h i_r - m s_h + \ell r_h \\
\frac{de_h}{dt_h^*} &= (1-\xi)\beta s_h i_h + \alpha s_h i_r - \theta e_h - m e_h \\
\frac{di_h}{dt_h^*} &= \theta e_h - \kappa i_h - \varphi i_h - m i_h - \omega i_h \\
\frac{dr_h}{dt_h^*} &= \kappa i_h - \ell r_h - m r_h + \omega i_h \\
\frac{ds_r}{dt_h^*} &= 1-(1-\eta)\alpha s_r i_r - n s_r \\
\frac{di_r}{dt_h^*} &= (1-\eta)\alpha s_r i_r - n i_r
\end{aligned}\right\} \quad (8)$$

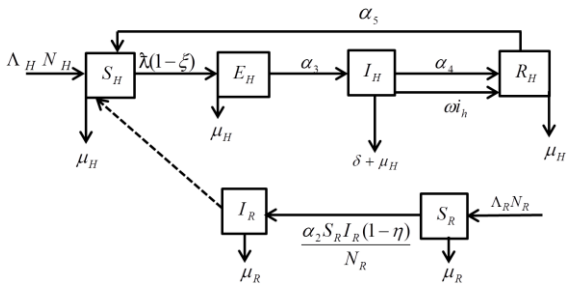

Figure 2. *Model diagram with control.*

4. Optimal Control Problem

The aim of the optimal control is to identify the control level which minimizes the number of exposed and infected classes. Therefore, we find the maximum value ξ^*, w^* and η^* of the controls ξ, ω and η at time t so that the trajectories which are combined to the states $s_h^*, e_h^*, i_h^*, r_h^*, s_r^*$ and i_r^* solve the above Lassa Fever model and $\xi^*(.)$, $\omega^*(.)$ and $\eta^*(.)$ minimizes the function defined below:

$$J = \min_{\varphi,w,r} \int_0^{t_f} e_h + i_h - \left(A_1\xi^2 + A_2\omega^2 + A_3\eta^2\right)dt \quad (9)$$

Subject to model equation (8).

Equation 9 above consist of the exposed and infected cases with the severity of the side effects, ξ, ω and η, of the control measures. Here, we employed $0 \le \xi < U$, $0 \le \omega < U$ and $0 \le \eta < U$, where $U = U_{\max}$. The conditions required for these optimal controls was obtained by the application of Pontraygin's Maximum Principle (PMP) is employed [14]. The Lagrangian is defined as following:

$$L = e_h + i_h - \left(A_1\xi^2 + A_2\omega^2 + A_3\eta^2\right) + \lambda_1 \frac{ds_h}{dt_h^*} + \lambda_2 \frac{de_h}{dt_h^*} + \lambda_3 \frac{di_h}{dt_h^*} + \lambda_4 \frac{dr_h}{dt_h^*} + \lambda_5 \frac{ds_r}{dt_h^*} + \lambda_6 \frac{di_r}{dt_h^*}$$

$$+c_{11}(b_1-\xi) + c_{12}(\xi-a_1) + c_{21}(b_2-\omega) + c_{22}(\omega-a_2) + c_{31}(b_3-\eta) + c_{32}(\eta-a_3)$$

Where c_{11}, c_{12}, c_{21}, c_{22}, c_{31}, $c_{32} \ge 0$ are penalty multipliers, which satisfy

$$c_{11}(b_1-\xi) = 0, \quad c_{12}(\xi-a_1) = 0 \quad \text{at } \xi^*$$

$$c_{21}(b_2-w) = 0, \quad c_{22}(w-a_2) = 0 \quad \text{at } w^*$$

$$c_{31}(b_3-\eta) = 0, \quad c_{32}(\eta-a_3) = 0 \quad \text{at } \eta^*.$$

Theorem 4: Given optimal controls ξ^*, ω^*, η^* and solution $s_h^*, e_h^*, i_h^*, r_h^*, s_r^*, i_r^*$ of the corresponding state system 8, there exist adjoint variables λ_i where $i = 1.2,...,6$

$$\lambda_1' = (\lambda_1-\lambda_2)(1-\xi)(\beta i_h + \alpha i_r) + \lambda_1 m$$

$$\lambda_2' = -1 + (\lambda_2-\lambda_3)\theta + \lambda_2 m$$

$$\lambda_3' = -1 + (\lambda_1-\lambda_2)(1-\xi)\beta s_h + \lambda_3(\kappa+\varphi+m+\omega) - \lambda_4(\kappa+\omega)$$

$$\lambda_4' = \lambda_4(\ell+m) - \lambda_1\ell$$

$$\lambda_5' = (\lambda_5-\lambda_6)(1-\eta)\alpha i_r + \lambda_5 n$$

$$\lambda_6' = (\lambda_1-\lambda_2)\alpha s_h + (\lambda_5-\lambda_6)(1-\eta)\alpha s_r + n\lambda_6$$

$\lambda_i(t_f) = 0$, transversality condition.

The optimality condition is given by

$$L_u = \frac{\partial L}{\partial \xi} = \frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial \eta} = 0.$$

Moreso, the optimal controls are defined as follow:

$$\xi^* = \min\left\{\max\left\{a_1, \frac{(\lambda_1-\lambda_2)\beta s_h i_h}{2A_1}\right\}, b_1\right\}$$

$$\eta^* = \min\left\{\max\left\{a_3, \frac{(\lambda_5-\lambda_6)\alpha s_r i_r}{2A_3}\right\}, b_3\right\}$$

$$\omega^* = \min\left\{\max\left\{a_2, \left(\frac{\lambda_5-\lambda_4}{2A_2}\right) i_h\right\}, b_2\right\}$$

Proof

The adjoint variables λ_i were solved in the system in the Lagrangian.

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial s_h}, \dot{\lambda}_2 = -\frac{\partial L}{\partial e_h}, \dot{\lambda}_3 = -\frac{\partial L}{\partial i_h}, \dot{\lambda}_4 = -\frac{\partial L}{\partial r_h}, \dot{\lambda}_5 = -\frac{\partial L}{\partial s_r}, \dot{\lambda}_6 = -\frac{\partial L}{\partial i_r}.$$

Thus,

$$\lambda_1' = (\lambda_1-\lambda_2)(1-\xi)(\beta i_h + \alpha i_r) + \lambda_1 m; \quad \lambda_1(\mathrm{t_f}) = 0$$

$$\lambda_2' = -1 + (\lambda_2-\lambda_3)\theta + \lambda_2 m; \quad \lambda_2(\mathrm{t_f}) = 0$$

$$\lambda_3' = -1 + (\lambda_1-\lambda_2)(1-\xi)\beta s_h + \lambda_3(\kappa+\varphi+m+\omega) - \lambda_4(\kappa+\omega); \quad \lambda_3(\mathrm{t_f}) = 0$$

$$\lambda_4' = \lambda_4(\ell+m) - \lambda_1\ell; \quad \lambda_4(\mathrm{t_f}) = 0$$

$$\lambda_5' = (\lambda_5-\lambda_6)(1-\eta)\alpha i_r + \lambda_5 n; \quad \lambda_5(\mathrm{t_f}) = 0$$

$$\lambda_6' = (\lambda_1-\lambda_2)\alpha s_h + (\lambda_5-\lambda_6)(1-\eta)\alpha s_r + n\lambda_6; \quad \lambda_6(\mathrm{t_f}) = 0$$

The optimal controls ξ^*, ω^*, η^* were resolved from $\frac{\partial L}{\partial \xi} = \frac{\partial L}{\partial \omega} = \frac{\partial L}{\partial \eta} = 0$, it follows that,

$$\frac{\partial L}{\partial \xi} = -2A_1\xi + (\lambda_1-\lambda_2)\beta s_h i_h - c_{11} + c_{12} = 0$$

$$\frac{\partial L}{\partial \omega} = -2A_2\omega + (\lambda_4-\lambda_3) i_h - c_{21} + c_{22} = 0$$

$$\frac{\partial L}{\partial \eta} = -2A_3\eta + (\lambda_5-\lambda_6)\alpha s_r i_r - c_{31} + c_{32} = 0$$

Therefore,

$$\xi = \frac{(\lambda_1-\lambda_2)\beta s_h i_h - c_{11} + c_{12}}{2A_1}$$

$$\omega = \frac{(\lambda_4-\lambda_3) i_h - c_{21} + c_{22}}{2A_2}$$

$$\eta = \frac{(\lambda_5-\lambda_6) s_r i_r - c_{31} + c_{32}}{2A_3}$$

There are 3-cases for the optimal controls ξ^*, ω^*, η^* respectively at time t.

Case I: $a_1 = \xi^*$ and $\xi \neq b_1$, $c_{11} = 0$.

$$a_1 = \xi = \frac{(\lambda_1-\lambda_2)\beta s_h i_h - c_{11} + c_{12}}{2A_1}$$

$$2A_1 a_1 - (\lambda_1-\lambda_2)\beta s_h i_h = c_{12} \geq 0$$

$$a_1 \geq \frac{(\lambda_1-\lambda_2)\beta s_h i_h}{2A_1}$$

Case II: $a_1 < \xi^* < b_1$, $c_{11} = c_{12} = 0$

$$\xi^* = \frac{(\lambda_1-\lambda_2)\beta s_h i_h}{2A_1}$$

Case III: $b_1 = \xi^*$, since $a_1 \neq \xi^*$, $c_{12} = 0$

$$b_1 = \xi^* = \frac{(\lambda_1-\lambda_2)\beta s_h i_h - c_{11} + c_{12}}{2A_1}$$

$$-2A_1 b_1 + (\lambda_1-\lambda_2)\beta s_h i_h = c_{11} \geq 0$$

$$b_1 \leq \frac{(\lambda_1-\lambda_2)\beta s_h i_h}{2A_1}$$

$$\xi^* = \begin{cases} a_1 & \text{if } L_{\xi^*} < a_1 \\ \dfrac{(\lambda_1 - \lambda_2)\beta s_h i_h}{2A_1} & \text{if } L_{\xi^*} = a_1 \\ b_1 & \text{if } L_{\xi^*} > a_1 \end{cases}$$

Following a similar argument for w^* and η^*, it follows that

$$\omega^* = \begin{cases} a_2 & \text{if } L_{\omega^*} < a_2 \\ \dfrac{(\lambda_4 - \lambda_3) i_h}{2A_2} & \text{if } L_{\omega^*} = a_2 \\ b_2 & \text{if } L_{\omega^*} > a_2 \end{cases}$$

$$\eta^* = \begin{cases} a_3 & \text{if } L_{\eta^*} < a_3 \\ \dfrac{(\lambda_5 - \lambda_6)\alpha s_r i_r}{2A_3} & \text{if } L_{\eta^*} = a_3 \\ b_3 & \text{if } L_{\eta^*} > a_3 \end{cases}$$

Therefore, it is summarized as follow in compact form.

$$\xi^* = \min\left\{\max\left\{a_1, \frac{(\lambda_1 - \lambda_2)\beta s_h i_h}{2A_1}\right\}, b_1\right\}$$

$$w^* = \min\left\{\max\left\{a_2, \left(\frac{\lambda_4 - \lambda_3}{2A_2}\right) i_h\right\}, b_2\right\}$$

$$\eta^* = \min\left\{\max\left\{a_3, \frac{(\lambda_5 - \lambda_6)\alpha s_r i_r}{2A_3}\right\}, b_3\right\}$$

Table 2. *Parameters Values.*

Parameters	Range	Reference	Scale Parameters	Values
Λ_H	1000*0.0003465	[17]	β	0.069-0.101
Λ_R	0.05	[20]	α	0.063-0.12
α_1	0.022-0.27	[17]	m	0.001
α_2	0.024-0.048	[17]	ℓ	0.4329
α_3	0.333	Assumed	θ	0.961
α_4	0.333-0.8	[19]	κ	0.0095
α_5	0.00385	[19]	φ	0.00056
δ	0.00019231	[18]	n	0.12821
μ_H	0.0003465	[18]		
μ_R	0.00641026	[19]		

5. Numerical Simulations

The numerical simulation of the model is performed in other to examine the sequel of Lassa Fever parameters in the growth of the virus. The numerical values in Table 2 and the previous states $s\_h(0) = 0.82$, $e\_h(0) = 0.08$, $c\_h(0) = 0.06$, $r\_h(0) = 0.04$, $s\_r(0) = 0.838$, and $c\_r(0) = 0.16$ were used. The graph below shows the Model simulation for some period of time. A numerical approach known as the Forward-Backward Sweep method was used to enable numerical modeling of the state and adjoint equations, and MATLAB script was to iteratively update the control by implementing the fourth-order Runge-Kutta method. This state is repeated until successive iterations are sufficiently close to each other [21, 22]. In this paper, the proposed control measures for three numerical modeling strategies of Lassa Fever model are summarized as follow:

(i) Strategy A: $u_1 = u_2 = u_3 = 0$

(ii) Strategy B: $u_1 = 0.2,\ u_2 = 0.3,\ u_3 = 0.2$

(iii) Strategy C: $u_1 = 0.5,\ u_2 = 0.65,\ u_3 = 0.45$

Where $\xi = u_1,\ \omega = u_2,\ \eta = u_3$

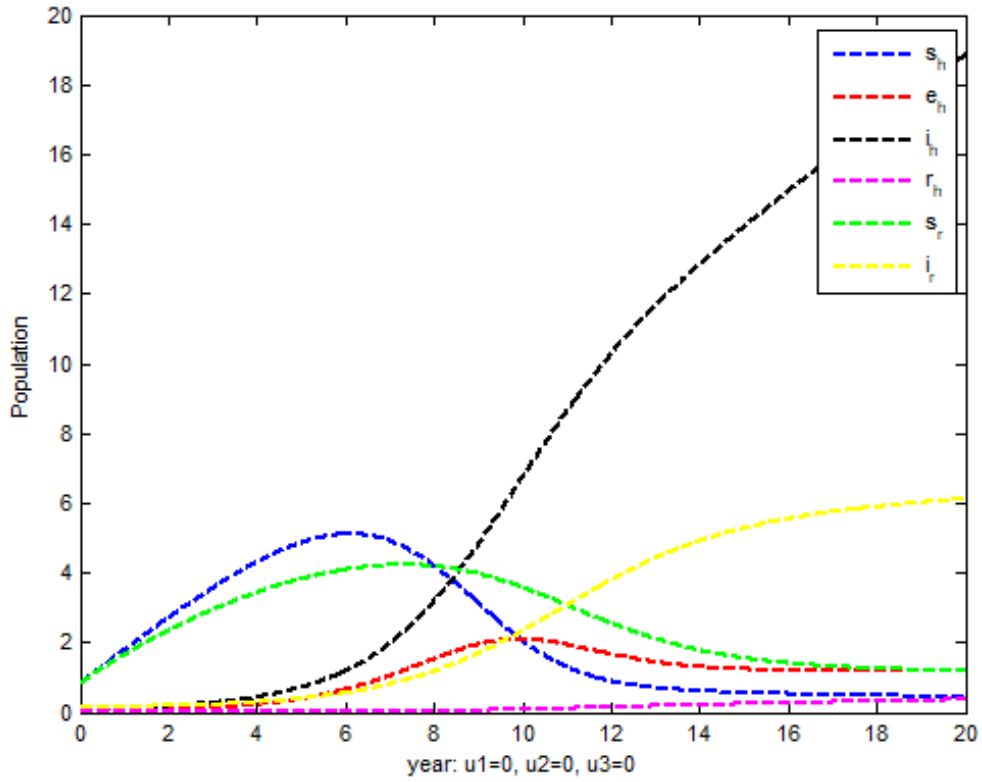

Figure 3. *Model Simulation for Strategy A.*

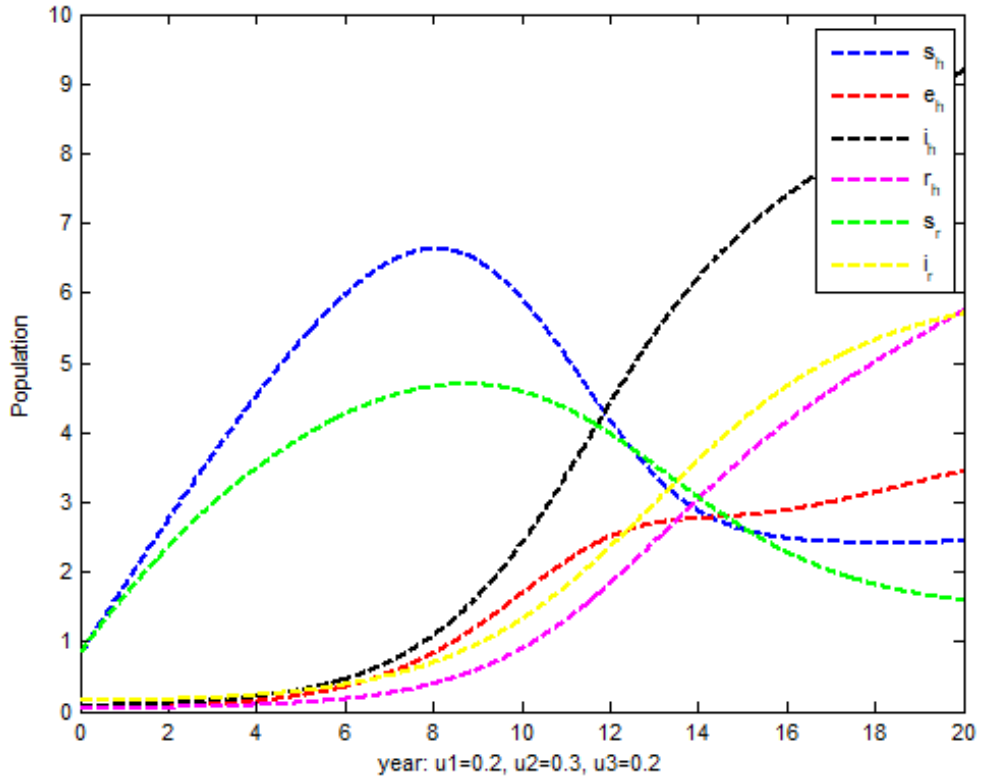

Figure 4. *Model Simulation for Strategy B.*

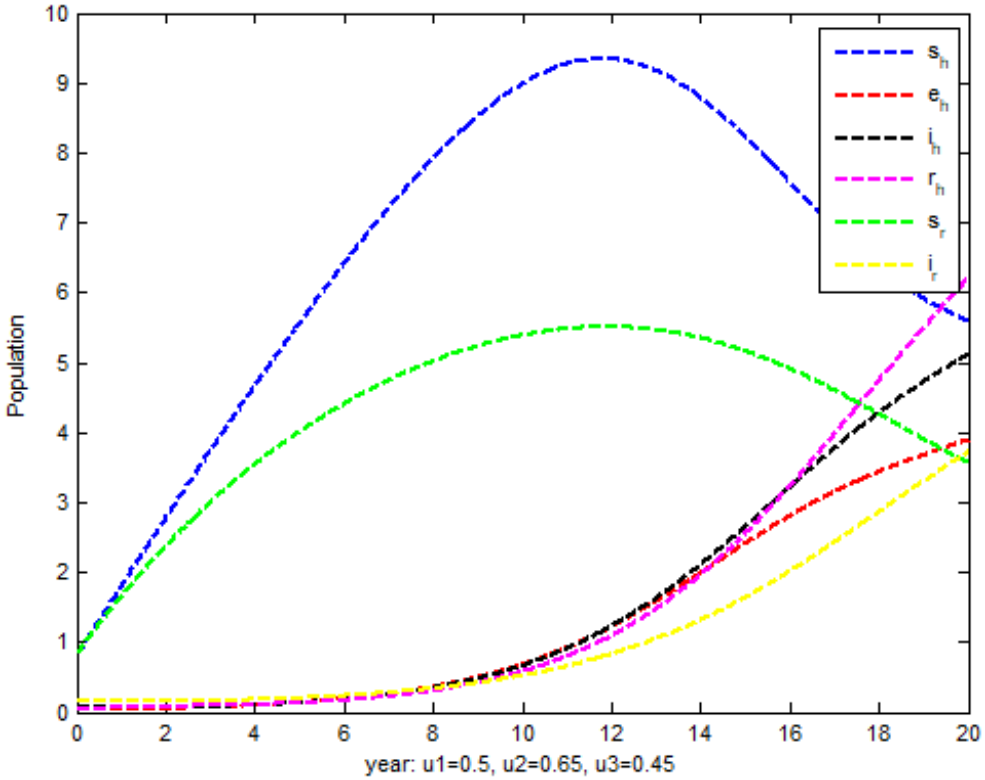

Figure 5. *Model Simulation for Strategy C.*

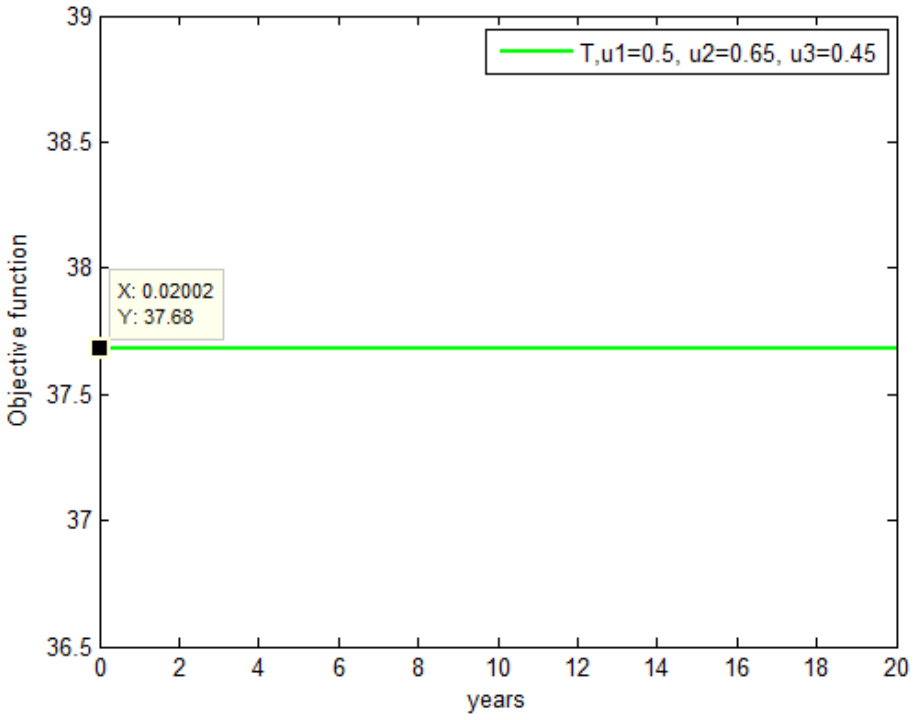

Figure 6. *The value of the objective function at a given maximum control level.*

Figures 3-5 above shows at a strategic level the effectiveness of the control measures in terms of the number of susceptible human, exposed human, infected human, recovered human, susceptible rodent and infected rodents populations.

T denote the objective function trajectory when $u_{1\max} = 0.5$, $u_{2\max} = 0.65$ and $u_{3\max} = 0.45$ respectively.

6. Conclusion

This work conducts a theoretical study on the control of model of Lassa Fever dynamics in Ebonyi State, Nigeria. The infection free equilibrium solution appears to be both locally and globally stable. The introduction of favourable conditions will lead to the suppression of the Lassa Fever and reduce the prevalence of Lassa Feever in Nigeria if the proposed controls are implemented. Model simulation show that the level of control increases, the number infected individuals decreases and the population of the recovered individuals increases. It soon became clear that there was a significant increase from strategy A to strategy C. Therefore, the percentage of individuals complying with the degree and level of care recovered in this task should be interpreted and used with caution. In any case, since multiple routes of infection are likely to exist, intervention strategies must become more context-specific. A more holistic approach to rodent control is needed, using effective and targeted control methods while maintaining a clean environment in homes and communities as adopted in this paper. To reduce secondary transmission of Lassa fever, additional education on personal hygiene and access to health facilities during illness is needed.

Abbreviations

LFFE: Lassa Fever-free equilibrium
IVP: Initial Value Problem

Conflicts of Interest

The authors declare no conflicts of interest.

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