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Research Article

# Modeling of the Complexity Propagation of Crack in a Ductile Material Under Complex Solicitation in Crack Tip: Introduction of a Matrix of Stress Intensity Factors of Bifurcation

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## Abstract

In fracture and damage mechanics, modeling of crack propagation has always been a source of difficulties. Numerous works have been carried out on this case at the crack tip, introducing new parameters: the Stress Intensity Factor (K); which is the local Irwin parameter, and also the Rice integral (J), the Griffith's energizing method, in which J and G are the global parameters around the crack tip. The problem of the crack remains very complex and difficult problem to be solved. Several methods are used to investigate the crack problem, namely the method of gradient, the numerical methods by finite elements, as well as the thermodynamic approach and the classical methods of Irwin, Griffith or Rice, according to the Intensity Stress Factor. This study adds to the work already carried out. Using the analytical analysis method of equations, we manage to show that the Stress Intensity Factor has a matrix of rank 3 at the crack tip, which is a better form since it includes complex combination cases of crack mode and bifurcation. Furthermore, when the material is subjected to complex stress, after analysis we emerge from a new singularity in (r) which is different from the classical mode. Finally, we are shown the new form of singularity, which is frequency dependent. This work can explain many situations, for example, the case of certain structural disasters showing the presence of cracks for complex or uncontrollable stress.

## Keywords

Crack, Matrix, Factor of Constraint Intensity (FIC), Frequency, Singularity

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# 1. Introduction

Since Irwin, [1, 10] some works have focused on the dynamic crack, highlighting the singularity of the stress intensity factor (FIK), at the crack tip. Considering the three elementary crack modes [1]: mode I, mode II and mode III, which respectively associated the FIK: KI, KII and KIII. The resolution of the problem being complex, several methods have been found in addition to the classical methods of Irwin, Rice or Griffith. Among these methods we can cite: numerical methods by finite elements see [12], gradient models [3], as well as the thermodynamic approach [10], the advantage of modeling a crack by a notch [5], and the experimental method [7], to close this literature we can mention works on the propagation of the crack with a law of Paris improven [6].

This difficulty is due to the presence of crack which transforms the domain around the crack into a plastic domain, which is governed by nonlinear equations.

The main objective of the present work is the investigation of complexity propagation of crack in a ductile material under complex solicitation in crack tip. On the contrary to other works on the calculation of the FIC in separate mode by finite elements [8], or in dynamics [2, 3, 11], which doesn't explicitly show the combination of the crack mode in complex cases, our work goes beyond by since in the case of a complex stress, leading to bifurcations, we introduce for the analytical analysis method, the notion of intensity factor matrix of stresses at the bottom of the crack, which for our knowledge is new. In addition we highlight the new singularity and the influence of frequencies on the singularity parameter. This work is a continuation of the work carried out in previous articles [13], which shows the new form of singularity at the crack tip.

# 2. Problem Formulations

## 2.1. Model Description

Let us consider a material with the initial mode I crack, submitted to a vibrating force ($F_i$), applied on the boundary surface ($\partial\Omega^d$) and the volume ($\Omega$), (see Figure 1).

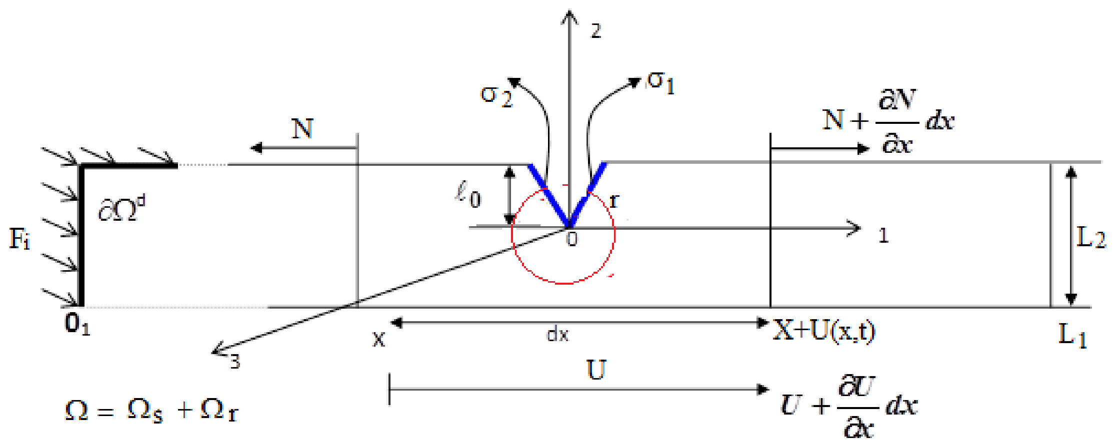


***Figure 1.*** *Material with a crack under complex solicitation in which U is the displacement, N is the component of $F_i$ in the $i^{th}$ direction, $\sigma_1$ and $\sigma_2$ are lips of the crack.*

The general equation governing this domain is given by the following relation

$$(\lambda + \mu) grad\ divU + \mu \Delta U = \frac{\partial^2 U}{\partial t^2} \tag{1}$$

By using the Clebsch theorem, the displacement vector U as well as all field of vector can be rewritten as

$$U = U^1 + U^2, \tag{2}$$

in which the $U_1$ is the derivative of a potential scalar and $U_2$ a potential vector. U is class $C^2$, which leads after the transformation to:

$$rotU^1 = 0\ and\ divU^2 = 0, \tag{3}$$

while taking into account Eq. (2) into Eq. (1), one has:

$$(\lambda + \mu) grad\ divU_1 + (\lambda + \mu) grad\ divU_2 + \mu \Delta U_1 + \mu \Delta U_2 = \frac{\partial^2 (U_1 + U_2)}{\partial t^2} \tag{4}$$

which accounting to Eq. (3), $with\ \Delta U = grad\ divU$ and $\Delta U_2 = -rot\ rotU_2$, leads to

$$(\lambda + \mu) \Delta U_1 + \mu \Delta U_2 + \rho \frac{\partial^2 (U_1 + U_2)}{\partial t^2} = 0 \tag{5}$$

### 2.2. Model Equation in Curvilinear Coordinates

Take into account the influence of the benchmark linked to the evolutionary trajectory of the crack, we make in the vicinity of the crack tip, the following changes of variable:

$$X = x + x_0, Y = y + y_0, Z = z + z_0, \tag{6}$$

where ($x_0$, $y_0$, $z_0$) are the Cartesian coordinates of the tip in the reference mark., (x, y, z) are the coordinates of the point located in bottom of crack or to the neighborhood of the crack tip so that, in the reference mark bound to the crack which can be expressed in spherical coordinated as:

$$X = rcos\varphi cos\theta, Y = rsin\varphi cos\theta, Z = rsin\theta, \tag{7}$$

leading the derivation operator to [10]

$$\Delta_{(x,y,z)} = \begin{bmatrix} \frac{\partial^2}{\partial X^2} - 2s_1 \frac{\partial^2}{\partial X \partial \ell} + s_1^2 \frac{\partial^2}{\partial \ell^2} \\ \frac{\partial^2}{\partial Y^2} - 2s_2 \frac{\partial^2}{\partial Y \partial \ell} + s_2^2 \frac{\partial^2}{\partial \ell^2} \\ \frac{\partial^2}{\partial Z^2} - 2s_3 \frac{\partial^2}{\partial Z \partial \ell} + s_3^2 \frac{\partial^2}{\partial \ell^2} \end{bmatrix} \tag{8}$$

with $s^2 = s_1^2 + s_2^2 + s_3^2$ since one knows that the trajectory ($\ell$) is function of ($x_0$, $y_0$, $z_0$) by the slant of the applied constraint ($\sigma^{app}$) in the crack tip. We obtain the new form of partial derivative:

$$\Delta_{(x,y,z)} = \left( \overbrace{\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}}^{\Delta(r,\varphi,\theta)} \right) + s^2 \frac{\partial^2}{\partial \ell^2} - 2\left( s_1 \frac{\partial^2}{\partial X \partial \ell} + s_2 \frac{\partial^2}{\partial Y \partial \ell} + s_3 \frac{\partial^2}{\partial Z \partial \ell} \right). \tag{9}$$

Considering elementary displacements dr, rdθ, rsinθdφ, the Laplacian $\Delta_{(x,y,z)}$ becomes in spherical coordinates:

$$\Delta_{(x,y,z)} = \Delta_{(r,\varphi,\theta)} + s^2 \frac{\partial^2}{\partial \ell^2} - 2\left( s_1 \frac{\partial^2}{\partial r \partial \ell} + s_2 \frac{\partial^2}{r \partial \theta \partial \ell} + s_3 \frac{\partial^2}{r \sin\theta \partial \varphi \partial \ell} \right) \tag{10}$$

leading to

$$\Delta_{(x,y,z)} = \Delta_{(r,\varphi,\theta)} + \widetilde{\Delta}_{(r\varphi,\theta,\ell)} \tag{11}$$

Where $\widetilde{\Delta}_{(r\varphi,\theta,\ell)}$ represents the "singular" Laplacian because it depends on the evolution of the crack tip, so that

$$\widetilde{\Delta}_{(r\varphi,\theta,\ell)} = s^2 \frac{\partial^2}{\partial \ell^2} - 2\left( s_1 \frac{\partial^2}{\partial r \partial \ell} + s_2 \frac{\partial^2}{r \partial \theta \partial \ell} + s_3 \frac{\partial^2}{r \sin\theta \partial \varphi \partial \ell} \right) \tag{12}$$

While applying this change of variable and reference mark to the resolution of the equation (5), we obtain in the spherical coordinates $(with\ a_1{}^2 = \frac{\mu}{\lambda}\ and\ a_2{}^2 = \frac{\rho}{\lambda})$

$$\Delta_{(r,\varphi,\theta)}\ U_1 + a_1^2\ \Delta_{(r,\varphi,\theta)}\ U_1 + \widetilde{\Delta}_{(r\varphi,\theta,\ell)}\ U_1 + a_1^2\ \widetilde{\Delta}_{(r\varphi,\theta,\ell)}\ U = a_2^2 \frac{\partial^2 U}{\partial t^2} \tag{13}$$

which is the general equation of propagation associated to the stationary reference mark, therefore the solution is a superposition of regular solutions $U^r$ and singular solution $U^\varepsilon$. To find $U^\varepsilon$, we consider the neighborhood domain of the crack tip who bound to the dynamic reference mark were origin is crack tip. The regulars terms bound to the stationary reference mark and out of the singular zone are both equal zero, leading Eq. (13) to

$$\widetilde{\Delta}_{(r\varphi,\theta,\ell)}\ U_1 + a_1^2\ \widetilde{\Delta}_{(r\varphi,\theta,\ell)}\ U = a_2^2 \frac{\partial^2 U}{\partial t^2} \tag{14}$$

This equation is the vibration equation and the propagation of the crack tip in the singular domain bound to the presence of the crack by the variable ℓ (length of the crack).

## 3. Structure of Solution

### 3.1. Preliminary

To determine the solution in the domain of crack or singularity zone who is characterized by the presence of the crack in this volume $\Omega_s$ and the rayon r limited by Γ. We introduce the criteria of bifurcation by the matrix [K] (Matrix of the Factor of constraint intensity defined as.

$$[K] = \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}, \tag{15}$$

which is the matrix of the FIC in the crack tip. Remembering the D. Leguillon and E. Sanchez-Palencia [13] works, we deduct the form of the solution $U_i^\varepsilon$ in the crack zone as:

$$U_i^\varepsilon = [K] \sum_\alpha \psi_\alpha\ R^\lambda(r)\ \underline{U}(\varphi,\theta,t) \tag{16}$$

in which λ is the degree of the singularity and $\underline{U}$ the effectif displacement in the crack tip. The component of matrix [K] is noted by $[k_{ij}] = [H_i(\omega)](K_j)$, were ($K_J$) is the factor of constraint intensity. While replacing [K] by his expression in (16):

$$U_i^\varepsilon = [H_i(\omega)](K_j), \sum_\alpha \psi_\alpha\ R^\lambda(r)\ \underline{U}(\varphi,\theta,t). \tag{17}$$

$H_i(\omega)$ is the applied strength compatible with the mode of crack. Considering the principle of separation of Fourier variables, we obtain

$$U_i^\varepsilon = [H_i(\omega)](K_j), \sum_\alpha \psi_\alpha(\varepsilon). R^\lambda(r). \Phi(\varphi). \Theta(\theta). e^{i\omega t} \tag{18}$$

Let $[\Upsilon]_\alpha = [H_i(\omega)](K_j), \sum_\alpha \psi_\alpha(\varepsilon)$. Eq. (18) become

$$U_i^{\varepsilon} = [\Upsilon]_{\alpha} R^{\lambda}(r).\Phi(\varphi).\Theta(\theta).e^{i\omega t}, \quad (19)$$

$$U_i^{\varepsilon} = U_{1i} + U_{2i} = [\Upsilon]_{\alpha} R^{\lambda}(r).\underline{U_i}(\varphi,\theta)e^{i\omega t}. \quad (20)$$

So that $U_{1i} = [\Upsilon]_{\alpha} R_1^{\lambda}(r).\Phi_1(\varphi).\Theta_1(\theta)e^{i\omega t}$ and $U_{2i} = [\Upsilon]_{\alpha} R_2^{\lambda}(r).\Phi_2(\varphi).\Theta_2(\theta)e^{i\omega t}$

We consider (14) with the operator $\widetilde{\Delta}_{(r\varphi,\theta,\ell)}$, the singular solution applied to the displacement is given by (18).

### 3.2. The Boundary Conditions on Side

$U = 0$ on $\partial\Omega \cup \sum_1 \cup \sum_2$ and for $r \to 0$. We have $U^{\varepsilon} = U^r$ on ($U^r$ is the regular displacement), on the $\partial\Omega^d$ $\sigma.n = F_i$ at t=o we suppose $U^{\varepsilon} = 0$, either $t = t_k$ and $\omega \neq 0$ we have $F(t_k) = 0$. In 3D the function R(r) is the form $r^{\lambda}$ with $\lambda > -(1/2)$.

### 3.3. Quasi-Static Approach

To solve (14), we are make the hypothesis of small perturbation (HPP) and consider the harmonic motion with the limit condition by $\frac{\partial U}{\partial x_i} = F_{i\cos\omega t}$, who drive us, in the spherical coordinates, when we introduce (18) in (14), we obtain the system of differential equation. For the resolution, let's apply the hypothesis of a harmonic movement with the limits condition. What drives us to (14). Let's introduce (18) in (14), this transformation gives the system of differential:

$$\begin{cases} \ddot{T} + \lambda_0^2 a^2 T = 0, \\ \ddot{\Phi} + \lambda_{2n}^2 \Phi = 0, \\ \ddot{\Theta} sin^2\theta + \dot{\Theta}\frac{1}{2}sin2\theta + (\lambda_{1n} sin^2\theta + \lambda_{2n})\Theta = 0, \\ r^2\ddot{R} + 2r\dot{R} + \left(r^{2\lambda_{0n}^2} - \lambda_{1n}^2\right)R = 0. \end{cases} \quad (21)$$

### 3.4. Discussion and Remarks

The system of equations (21) translates through the two last lines the presence of the singularity and the weakly (amortize) at the crack tip by the terms: $(sin^2\theta, r^2)$, $(\frac{1}{2}\sin 2\theta,\ 2r)$ and $\lambda_{1n}sin^2\theta + \lambda_{2n}, r^{2\lambda_{0n}^2} - \lambda_{1n}^2$ respectively of Θ(θ) and Φ(φ). It can't be assimilated to terms of intertie's, amortizations and stiffness. The condition of the weakly (amortize) imposes the relations: $(\frac{1}{2}\sin 2\theta < 2sin^2\theta,\ \frac{1}{2}\sin 2\theta < 2(\lambda_{1n}sin^2\theta + \lambda_{2n})$, and $(2r < r^2,\ 2r < (\lambda_{0n}r^2 + \lambda_{1n})$ where

$$\begin{cases} r > 2, \\ \lambda_{0n}\, r^2 - 2r - \lambda_{1n} > 0, \end{cases} \quad and \quad \begin{cases} \tan\theta > 1, \\ 2\lambda_{1n}sin^2\theta - sin2\theta + \lambda_{2n} > 0. \end{cases} \quad (22)$$

Then, from inequality (22) we obtain $r \in ]2, r_0[\cup]r_1, +\infty[$, $r_0 = \frac{1-\sqrt{1+\lambda_{0n}\lambda_{1n}}}{\lambda_{0n}}$ and $r_1 = \frac{1+\sqrt{1+\lambda_{0n}\lambda_{1n}}}{\lambda_{0n}}$ and $\lambda_0 \in ]\frac{4+\lambda 1n}{4}, +\infty[$the system is not harmonic and the crack is static.

$r \in ]r_0, r_1[$, the system is harmonic.

Finally the analysis of Eq. (22) shows that $\theta \in ]\frac{\pi}{4}, \frac{5\pi}{4}[$ for the non-harmonic case, and $\theta \notin ]\frac{\pi}{4}, \frac{5\pi}{4}[$ for the harmonic case.

## 4. Analytical Solutions

### 4.1. Case of the First Line of Eq. (21)

The form of its solution is $T_{\alpha}(t) = B_{0\alpha\lambda_{0\alpha}}(\omega)cos[\omega_{0n}t + \beta_{0\alpha\lambda_{0\alpha}}(\omega)]$, with $\omega_{0n}^2 = \lambda_0^2 a^2$ and $B_{0\alpha\lambda_{0\alpha}}(\omega)$ the phase difference between the excitation and the answer of the material. After application of the boundary condition, we obtain $(\omega_{0n} = (2n+1)\omega_n)$

$$T_{\alpha}(t) = B_{0\alpha}\lambda_{0\alpha}c(\omega)cos[(2n+1)\omega_n], \lambda_{0\alpha} = \frac{(2n+1)\omega_n}{a_{\alpha}} \quad (23)$$

and

$$U_i^{\varepsilon} = [\Upsilon]_{\alpha} \sum_{n=0}^{+\infty} R_{\sigma n}(r).\Phi_{\alpha n}(\varphi).\Theta_{\alpha n}(\theta)cos[(2n+1)\omega_n t] \quad (24)$$

### 4.2. Case of the Second Line of Eq. (21)

The solution is

$$\Phi_{\alpha n} = B_{5\alpha n}exp\left(\sqrt{-\lambda_{2\alpha}(n,\omega)}\varphi\right) + D_{1\alpha n}exp\left(-\sqrt{-\lambda_{2\alpha}(n,\omega)}\varphi\right) \quad (25)$$

Where

$$\Phi_{\alpha n} = B_{5\alpha n}cos\left(\left(\sqrt{\lambda_{2\alpha}(n,\omega)}\right)\varphi\right) + D_{1\alpha n}sin\left(\left(\sqrt{\lambda_{2\alpha}(n,\omega)}\right)\varphi\right) \quad (26)$$

corresponding respectively to $\lambda_{2\alpha}(n,\omega) < 0$ and to $\lambda_{2\alpha}(n,\omega) > 0$, wiht $\phi_{2\alpha}(n,\omega) = 0$ if to $\lambda_{2\alpha}(n,\omega) = 0$.

### 4.3. Case of the Last Line of Eq. (21)

The form of solution this equation is given by the method of series by

$$R_{\alpha n}(r) = r^{P_{n\alpha}} \sum_{m=0}^{+\infty} A_{\alpha mn}^{(j)} r^m \tag{27}$$

the determinant equation of the last line of Eq. (21) is $P^2 + P - \lambda_1 n = 0$, the solution of is

$$P_{1n\alpha} = \frac{-1+\sqrt{1+4\lambda_{1\alpha}(n,\omega)}}{2} \ and \ P_{2n\alpha} = \frac{-1-\sqrt{1+4\lambda_{1\alpha}(n,\omega)}}{2} \tag{28}$$

Where $|P_{1n} - P_{2n}| = \sqrt{\delta_n}$ with $\delta_n = 1 + 4\lambda_{1\alpha}(n,\omega)$ or $\delta_n = \Gamma_{n\alpha} e^{iJ_{n\alpha}}$, where $\Gamma_{n\alpha}$ is the module of $\delta_n$, and $J_{n\alpha}$ the argument of $\delta_n$ ($i^2 = -1$).

$$A_{\alpha mn}^{(j)} = \frac{-(2n+1)\omega A_{\alpha(m-1)n}}{a\left[(m+P_{Jn\alpha})^2+(m+P_{Jn\alpha})-\lambda_{1\alpha}\right]} \tag{29}$$

the recurrent relation between $A_{\alpha mn}$ and $A_{\alpha(m-2)n}$, with $(m + Pjn\alpha)^2 + (m + Pjn\alpha) - \lambda_{1\alpha}(n,\omega) \neq 0$, $m \geq 2$ $and\ j = \{1,2\}$, for j = 1 we have $P_{1n\alpha}$, for j = 2 we have $P_{2n\alpha}$.

### 4.4. Analysis and Interpretation

For $\sqrt{\delta_n} \notin N^*$

$K_n' \notin N^*$ it exists $\lambda_1 n$ so that $\sqrt{1 + 4\lambda_{1n}} = K_n'$ where $\lambda_{1n} = \frac{K_n'^2-1}{4}$, the solution is

$$R_{\alpha n\alpha}(r, P_{1n\alpha}, P_{2n\alpha}) = \begin{cases} Real\left(B_{1n\alpha}R_{1n\alpha}(r) + B_{2n\alpha}R_{2n\alpha}(r)\right) \\ + \\ Im\left(B_{1n\alpha}R_{1n\alpha}(r) + B_{2n\alpha}R_{2n\alpha}(r)\right) \end{cases} \tag{30}$$

where $B_{1n\alpha}$ and $B_{2n\alpha}$ are the constants.

$$R_{1n\alpha}(r) = \begin{cases} \left[r^{\frac{-1}{2}+\frac{\sqrt{\Gamma n\alpha}}{2}\cos\frac{Jn\alpha}{2}} \sum_{m=0}^{+\infty} A_{\alpha mn}^{(1)} r^m\right], \\ \times \\ \cos\left(\frac{\sqrt{\Gamma n\alpha}}{2}\left(\sin\left(\frac{Jn\alpha}{2}\right)\right) ln(r)\right) + i\sin\left(\frac{\sqrt{\Gamma n\alpha}}{2}\left(\sin\left(\frac{Jn\alpha}{2}\right)\right) ln(r)\right). \end{cases} \tag{31}$$

$$R_{2n\alpha}(r) = \begin{cases} \left[r^{\frac{-1}{2}+\frac{\sqrt{\Gamma n\alpha}}{2}\cos\frac{Jn\alpha}{2}} \sum_{m=0}^{+\infty} A_{\alpha mn}^{(2)} r^m\right], \\ \times \\ \cos\left(\frac{\sqrt{\Gamma n\alpha}}{2}\left(\sin\left(\frac{Jn\alpha}{2}\right)\right) ln(r)\right) - i\sin\left(\frac{\sqrt{\Gamma n\alpha}}{2}\left(\sin\left(\frac{Jn\alpha}{2}\right)\right) ln(r)\right). \end{cases} \tag{32}$$

For $\sqrt{\delta_n} \in \mathbb{N}$, one has: $\sqrt{1 + 4\lambda_{1n}} = k_n$ leading to $\lambda_{1n} = \frac{K_n^2-1}{4}$, ($k_n \in N$) where

$$R_{\alpha nb}(r, P_{1n\alpha}, P_{2n\alpha}) = B_{3n\alpha}R_{3n\alpha}(r) + B_{4n\alpha}R_{4n\alpha}(r). \tag{33}$$

$B_{3n\alpha}$,, $B_{4n\alpha}$ are the constants.

$$R_{3n\alpha}(r) = r^{\frac{kn\alpha-1}{2}} \sum_{m=0}^{+\infty} A_{\alpha mn}^{(1)} r^m, \tag{34}$$

$$R_{4n\alpha}(r) = \mu_{n\alpha} r^{\frac{K_n^2-1}{4}} ln(r) \sum_{m=0}^{+\infty} A_{\alpha mn}^{(1)} r^m + r^{-\frac{K_n^2+1}{4}} \sum_{m=0}^{+\infty} A_{\alpha mn}^{(2)} r^m. \tag{35}$$

where

$$U_i^{\varepsilon} = [\Upsilon]_\alpha \begin{cases} \sum_{n\notin I_\alpha} B_{0\alpha n\alpha}\psi_{\alpha n\alpha}(\varepsilon)[R_\alpha(r, P_{1n\alpha}, P_{2n\alpha})\Phi_{\alpha n\alpha}(\varphi)\Theta_{\alpha n\alpha}(\theta)]cos[(2n+1)\omega_n t] \\ + \\ \sum_{n\in I_\alpha} B_{0\alpha nb}\psi_{\alpha nb}(\varepsilon)[R_b(r, P_{1n\alpha}, P_{2n\alpha})\Phi_{\alpha nb}(\varphi)\Theta_{\alpha nb}(\theta)]cos[(2n+1)\omega_n t] \end{cases} \tag{36}$$

which is the general solution.

According to [13] for the 3D case, the particularity of the characteristic of exposing of singularity on the crack tip is strictly greater than -1/2. While applying this condition to our results one has: $\frac{-1}{2} \pm \frac{\sqrt{\Gamma_{n\alpha}}}{2} \cos\frac{J_{n\alpha}}{2} > \frac{-1}{2}$ and $\frac{-1 \pm k_{n\alpha}}{2} > \frac{-1}{2}$ or $\pm\sqrt{\Gamma_{n\alpha}} \cos\frac{J_{n\alpha}}{2} > 0$ and $\pm k_n > 0$, finally one has $0 < J_{n\alpha} < 2\pi$.

By identification, we have the convergence condition of the solution $\sum_{m=0}^{+\infty} A_{\alpha mn}^{(J)} r^m = 1$ and $P_{n\alpha} > \frac{-1}{2}$ . where $\frac{-1 \pm \sqrt{1+4\lambda_{1\alpha}}}{2} > \frac{-1}{2}$ soit $\lambda_{1\alpha}(n,\omega) > \frac{-1}{4}$. The condition 3D impose $P_j = P_{1n\alpha}$ and $B_{2n\alpha} = B_{4n\alpha} = 0$, (31) and (34) becomes

$$R_{\alpha n a}(r, P_{1n\alpha}, P_{2n\alpha}) = B_{1n\alpha}(Real\ R_{1n\alpha}(r) + Im R_{1n\alpha}(r)),\quad R_{\alpha n a}(r, P_{1n\alpha}, P_{2n\alpha}) = B_{1n\alpha}\ R_{1n\alpha}(r),\quad R_{\alpha n b}(r, P_{1n\alpha}, P_{2n\alpha}) = B_{3n\alpha}\ R_{3n\alpha}(r), \tag{37}$$

and

$$R_{1n\alpha}(r) = r^{\frac{-1}{2} + \frac{\sqrt{\Gamma_{n\alpha}}}{2}\cos\frac{J_{n\alpha}}{2}} \sum_{m=0}^{+\infty} A_{\alpha mn}^{(1)} r^m expi\left(\frac{\sqrt{\Gamma_{n\alpha}}}{2}\sin\left(\frac{J_{n\alpha}}{2}\right)\right) ln(r),\quad R_{3n\alpha}(r) = r^{\frac{-1+k_{n\alpha}}{2}} \sum_{m=0}^{+\infty} A_{\alpha mn}^{(1)} r^m. \tag{38}$$

Finally, the solution must be written

$$U_i^{\varepsilon} = [\Upsilon]_\alpha \begin{cases} \sum_{n \notin I_\alpha} B_{01\alpha n a} \psi_{\alpha n a}(\varepsilon) [R_{1n\alpha}(r) \Phi_{\alpha n a}(\varphi) \Theta_{\alpha n a}(\theta)] cos[(2n+1)\omega t] \\ + \\ \sum_{n \in I_\alpha} B_{03\alpha n b} \psi_{\alpha n b}(\varepsilon) [R_{3n\alpha}(r) \Phi_{\alpha n b}(\varphi) \Theta_{\alpha n b}(\theta)] cos[(2n+1)\omega t] \end{cases} \tag{39}$$

This approach show the displacements are generally singular in $r^{\frac{-1+k_{n\alpha}}{2}}$ and $r^{\frac{-1}{2} + \frac{\sqrt{\Gamma_{n\alpha}}}{2}\cos\frac{J_{n\alpha}}{2}}$.

The modes $n \in N^*$ and $n \neq k$, (is the number of mode n who $F_i(t_k) = 0$, and $\omega_k = \frac{\pi}{2t_k}(2k+1)$, we have $\omega_{0n} = (2n+1)\omega_n$, ($\omega_{0n}$ is the real value of pulsation). Considering the relation between $\omega_{0n}$ and $\omega_n$.

Firstly, when $n > n_0 \rightarrow \omega_n 2\ [0, 1]$ and $n \leq n_0 \rightarrow \omega_n\ 2\ [0, 1]$ with $n_0 = E(\frac{\omega_{0n}}{2})$ -1, (E(X) = is the integer part of X), we obtain the graph who confirm this situation:

Figure 2 confirms the situation in the singular domain, when (n) increases the frequency supply towards zero, confirming that we are the restriction domain.

We noticed that, when n grow up, the frequencies $\omega_n$ decreases and offering toward zero. For n offering toward zero we have $\omega_n$ who increase. Secondly, we are fixed n≠0, when (n, $\omega_{0n}$) → +∞, and $\omega_n$ → 0, the crack under solicitation max, we have the propagation of the crack.

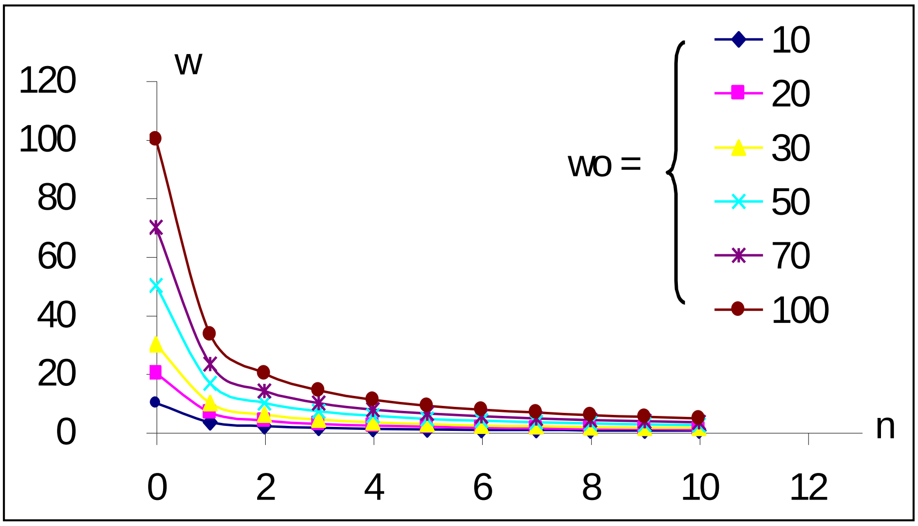


***Figure 2.*** *Graph of $\omega_n$ function according to n for different values of $\omega_0$.*

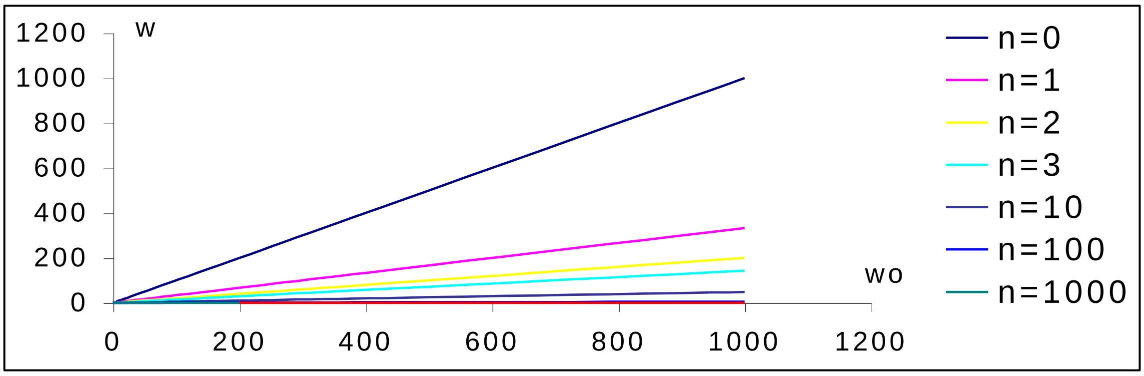


*Figure 3. Graph of $\omega_n$ function of $\omega_{0n}$. n represent the mode of vibration or frequencies.*

$\omega_{0n}$ is the eigenfrequencies.

To warn the reduction of the propagation crack is better to choose one material who $\omega_{0n}$ is big. Besides, this graph (Figure 3) shows that the Eigen-frequency of the material change, confirming the influence of damages on a characteristics of the material [11, 13].

# 5. Singularity Equation of Frequencies

## 5.1. Resolution for (14)

In the crack tip, the form of displacements is

$$U = [K]e^{i\omega t} r^{\lambda_n} U(\theta,\varphi), with\ \lambda_n > -\tfrac{1}{2}. \tag{40}$$

Considering the equation (21) and the separating of the solution $U(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$, applying (14), one has

$$\frac{\partial^2 U}{\partial t^2} = -\omega^2 [K] e^{i\omega t} r^{\lambda_n} \Theta(\theta)\Phi(\varphi) \tag{41}$$

and

$$\Delta_{(r,\theta,\varphi)} U = [K]e^{i\omega t} \begin{cases} \ddot{R}\Theta\Phi + \frac{2}{r}\dot{R}\Theta\Phi + \frac{1}{r^2 sin^2\theta}\ddot{\Phi}R\Theta, \\ +\frac{1}{r^2}\ddot{\Theta}R\Phi + \frac{1}{r^2 tan^2\theta}\dot{R}\dot{\Theta}\Phi. \end{cases} \tag{42}$$

Considering the domain around the crack tip or the singular domain we have:

$$\widetilde{\Delta}_{(r\varphi,\theta,\ell)} = \left[ s^{2\frac{\partial^2[K]}{\partial \ell^2}} - 2\left( s_1 \frac{\partial^2[K]}{\partial r \partial \ell} + s_2 \frac{\partial^2[K]}{r\partial\theta\partial\ell} + s_3 \frac{\partial^2[K]}{r\sin\theta\partial\varphi\partial\ell} \right) \right] \times r^{\lambda_n} e^{i\omega t}\Theta\Phi \tag{43}$$

and (21) become

$$-\omega^2 = a^2 \left[ \left(\frac{\ddot{R}}{R}\right) + \frac{2}{r}\left(\frac{\dot{R}}{R}\right) + \frac{1}{r^2 sin^2\theta}\left(\frac{\ddot{\Phi}}{\Phi}\right) + \left( \frac{1}{r}\frac{\ddot{\Theta}}{\Theta} + \frac{1}{r^2 tan\theta}\frac{\dot{\Theta}}{\Theta} \right) \right]. \tag{44}$$

Knowing that $R(r) = r^{\lambda_n}, \dot{R}(r) = \lambda_n r^{\lambda_n - 1}, \ddot{R}(r) = \lambda_n(\lambda_n - 1)r^{\lambda_n - 2}$, either

$$\frac{\ddot{R}}{R} = \lambda_n(\lambda_n - 1)r^{-2},\ \frac{\dot{R}}{R} = \lambda_n r^{-1},\ \frac{\ddot{\Theta}}{\Theta} = \lambda_n. \tag{45}$$

Concerning $\ddot{\Theta} sin^2\theta + \dot{\Theta}\frac{1}{2} sin2\theta + (\lambda_{1n} sin^2\theta + \lambda_{2n})\Theta = 0$, to multiply by $\frac{1}{r^2 \Theta sin^2\theta}$ we obtain

$$\frac{1}{r^2}\frac{\ddot{\Theta}}{\Theta} + \frac{1}{r^2 tan\theta}\frac{\dot{\Theta}}{\Theta} = -\frac{1}{r^2 sin^2\theta}(\lambda_{1n} sin^2\theta + \lambda_{2n}) \tag{46}$$

Introduce (45), (46) in (44), we obtain the equation of the singularity of the frequencies

$$\omega_n^2 = \frac{a^2}{r^2}\left[ \frac{\lambda_{1n} sin^2\theta + \lambda_{2n}}{sin^2\theta} - \lambda_n(\lambda_n - 1) - 2\lambda_n - \frac{\lambda_{2n}}{sin^2\theta} \right] \tag{47}$$

$$\omega_n = \frac{a}{r}\sqrt{\lambda_{1n} - \lambda_n(\lambda_n - 1)}, \lambda_{1n} = \frac{k_n'^2 - 1}{4} \tag{48}$$

($k_n' \notin N^*$). The frequencies are function of the degree of singularity in crack tip and the domain of fissuration by the parameter r. For $\lambda_n(\lambda_n+1) = 0$ ($\lambda_n = 0$ or $\lambda_n = -1$), (53) becomes of frequency equation of non-evolution of crack $\omega_n = \frac{a}{r}\sqrt{\lambda_{1n}}$.

## 5.2. Interpretation

The dispersion relations (47) and (48) confirm the nonlinearity and disruptions at the level of frequencies to the neighborhood of the crack tip.

Figure 4 represents the graph of $\omega_n(r)$ with the parameter (a) function a variable (a) for the different values of ($\omega_n(r) =$

$\frac{1}{r}\sqrt{\lambda_{1n}}, \omega_n(r) = \frac{5}{r}\sqrt{\lambda_{1n}}, \omega_n(r) = \frac{10}{r}\sqrt{\lambda_{1n}}, \omega_n(r) = \frac{20}{r}\sqrt{\lambda_{1n}}, \omega_n(r) = \frac{50}{r}\sqrt{\lambda_{1n}}$). It gives us a good interpretation for the evolution of frequencies: This graph shows that when r is big the frequency decreases explaining the fact that one is far from of the crack tip, on the other hand, when one comes closer of the crack tip ($r = 0$), we are the plastic domain and greatly nonlinear equation (21). In neighbor domain ($r = 0$), the presence of crack leads to strongly increasing of the frequency.

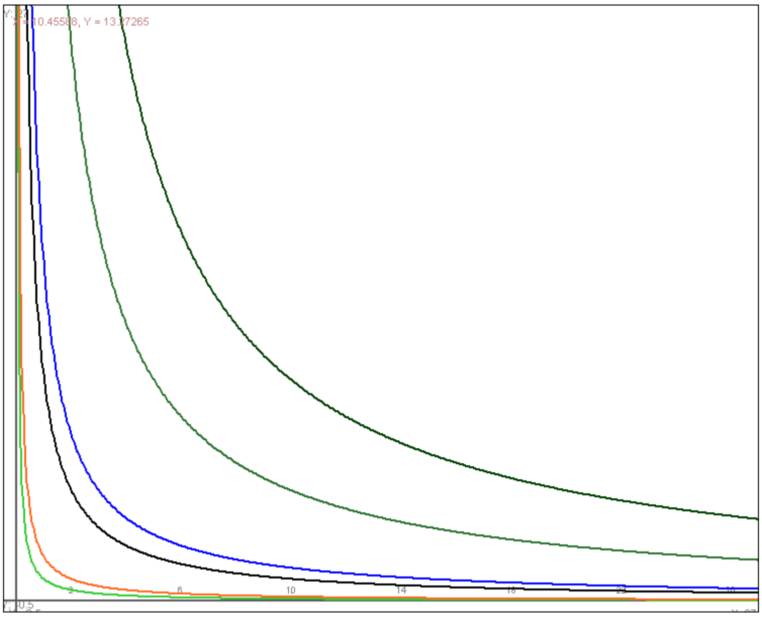

***Figure 4.*** *The graph of evolution the* $\omega_n = f(r)$ *according to the n.*

The different curves show that, for every value of the frequency, when r increases, far from crack $\omega_n$ offers toward zero: we are the linear domain.

## 6. Conclusion

Through this survey we putted in light three aspects: the first aspect concerning the FIK that can get therefore under matrix of rank 3, the eigenvalues of this matrix are the classical mode $K_I$, $K_{II}$ and $K_{III;}$ secondly, the appearing of new forms of singularity in the crack tip: third, the influence of frequencies on the singularity therefore on the propagation of the crack. The present work reveals that a vibratory loading of a material with an initial mode I of crack leads to the appearance of non classical singular modes of the type $r^{\Upsilon_n}\cos(\beta_n lnr + \chi_n)$, around the crack tip. These modes, in combination with those classically encountered in work on the calculation of the displacement field in materials exhibiting a crack, can produce a change in crack path as well as mode bifurcations and branching effects at the crack tip origin. The damage of the material is then more pronounced and happens more quickly than it does in absence of the vibratory loading. In addition, the way to perform the inner and the outer asymptotic expansions around the crack tips undergoes a substantial changing in some way due to by the complexity of the radial functions $R_{\alpha na}(r, P_{1n\alpha}, P_{2n\alpha})$ and $R_{\alpha nb}(r, P_{1n\alpha}, P_{2n\alpha})$. For $k_{n\alpha} = 0$ we obtain the classical case who the displacements are singular in $r^{\frac{-1}{2}}$. Beside for $k_{n\alpha} \neq 0$, the displacements are singular in $r^{\frac{-1+k_{n\alpha}}{2}}$, we have the small perturbation and the strong perturbation for the singularity of $r^{\frac{-1}{2}+\frac{\sqrt{\Gamma_{n\alpha}}}{2}}\cos\frac{J_{n\alpha}}{2}$.

## Conflicts of Interest

The authors declare no conflicts of interest.

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