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A New Second-order Maximum-principle-preserving Finite-volume Method for Flow Problems Involving Discontinuous Coefficients

Abdou Njifenjou[1, *], Abel Toudna Mansou[2], Moussa Sali[3]

[1]Department of Mathematics, National Advanced School of Engineering, University of Yaounde 1, Yaounde, Cameroon
[2]Department of Mathematics, Faculty of Science, University of Douala, Douala, Cameroon
[3]Department of Civil Engineering and Architecture, Ecole Nationale Superieure Polytechnique de Maroua, Maroua, Cameroon

Email address:

njifa2000@yahoo.fr (Abdou Njifenjou), abelmansou12s@yahoo.com (Abel Toudna Mansou), moussa_sali@yahoo.fr (Moussa Sali)
[*]Corresponding author

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Abstract: A new development of Finite Volumes (FV, for short) and its theoretical analysis are the purpose of this work. Recall that FV are known as powerful tools to address equations of conservation laws (mass, energy, momentum,...). Over the last two decades investigators have succeeded in putting in place a mathematical framework for the theoretical analysis of FV. A perfect illustration of this progress is the design and mathematical analysis of Discrete Duality Finite Volumes (DDFV, for short). We propose now a new class of DDFV for 2nd order elliptic equations involving discontinuous diffusion coefficients or nonlinearities. A one-dimensional linear elliptic equation is addressed here for illustrating the ideas behind our numerical strategy. The algebraic structure of the discrete system we have got is different from that of standard DDFV. The main novelty is that the so-called diamond mesh elements are confined in homogeneous zones for flow problems governed by piecewise constant coefficients. This is got from our judicious definition of the primal mesh. The gain is that there is no need to compute homogenized coefficients to be allocated to the so-called diamond cells as required to conventional DDFV. Notice that poor homogenized permeability allocated to diamond elements leads to poor approximations of fluxes across grid-block interfaces. Moreover for 1-D flow problems in a porous medium involving permeability discontinuities (piecewise constant permeability for instance) the proposed FV scheme leads to a symmetric positive-definite discrete system that meets the discrete maximum principle; we have shown its second order convergence under relevant assumptions.

Keywords: Diffusion-reaction Problems, New Finite Volume Scheme, Diffusion Coefficient Discontinuities, Second Order Convergence

1. Introduction

Let $\Omega =]a; b[$ be an interval from the set of real numbers $\mathbb{R}$, where $a, b \in \mathbb{R}$ with $a < b$ are given. Also given as data are real-valued functions λ, f and μ.

We are interested in investigating a new Finite Volume approximation of the solution u to the following diffusion-reaction problem:

Find u in an adequate function space such that:

$$-\left[\lambda(x)u'(x)\right]' + \mu(x)u(x) = f(x) \quad \text{in} \quad \Omega \tag{1}$$

$$u(a) = u(b) = 0. \tag{2}$$

It is well known that, in the context of discontinuous diffusion coefficients, the conventional Finite Volume solution to the preceding problem, without reaction term (i.e. with $\mu = 0$), displays only a first order convergence in L^2-norm, L^∞-norm and in some discrete energy norm (see Chapter 1 from the work of R. Eymard et al. which is considered as one of the most important references on this subject.

[21]). In the previous reference (see again Chapter 1) it is indicated that however second order convergence is achieved if the exact solution to (1)-(2), without reaction term, is a C^4-function, assuming a constant diffusion coefficient λ and a uniform mesh size on $\overline{\Omega}$. Higher space dimension problems of the type (1)-(2) are considered in the literature (see for instance the following references and references therein. $[21-23]$). Most numerical methods in general and conventional finite volumes in particular give only a first order convergence. However assuming that the diffusion matrix lies in $W^{1,\infty}(\Omega)$ in two research articles signed by Franco Brezzi and some of his colleagues a second order convergence has been got for a mimetic finite difference solution. $[22, 24]$. Similar results have been got by other investigators with conventional cell-centered finite volumes for the Laplace operator in rectangular domains. [21]. Let us mention an interesting result due to Pascal Omnes who has obtained a second-order convergence for a function reconstructed from a Finite Volume approximation of the 2D-Laplace operator on Delaunay-Voronoi meshes.[18].

Developing a new Finite Volume scheme based upon a 2nd order convergence technique on non-uniform meshes is our aim in this work. We will be addressing one dimensional flow problems of the type (1)-(2) in the context of discontinuous diffusion coefficients. We will proceed in such a way that the discrete problem involves discrete unknowns located at cell points and at vertex points of a primary coarse mesh. These points being cell centers of a finer mesh associated with the primary mesh. For obtaining the discrete equations, the numerical technique used is not that of the conventional Discrete Duality Finite Volumes as we will see later.

Main assumptions on the data:

1. On one hand the function λ is supposed to meet the following assumptions:

$$\begin{cases} \circ \quad \lambda(x) = \sum_{s=1}^{S} \lambda_s \mathbf{1}_{\mathcal{O}_s}(x) \quad \text{a.e.} \quad \text{in} \quad \Omega \\ \\ \circ \quad \exists \lambda_-, \lambda_+ \in \mathbb{R}_+^* \quad \text{such that} \quad \lambda_- \leq \lambda(x) \leq \lambda_+ \text{ a.e.} \quad \text{in} \quad \Omega, \end{cases} \tag{3}$$

where $\mathbb{R}_+^*$ stands for the subset of $\mathbb{R}$ made up of non-negative real numbers and where $\{\mathcal{O}_s\}$ is a partition of Ω made up of non-empty open subintervals of Ω, while $\{\mathbf{1}_{\mathcal{O}_s}\}$ is a family of functions defined almost everywhere (a.e. for short) in Ω as:

$$\mathbf{1}_{\mathcal{O}_s}(x) = \begin{cases} 1 \quad \text{if} \quad x \in Int(\mathcal{O}_s) = \mathcal{O}_s \\ \\ 0 \quad \text{if} \quad x \in Ext(\mathcal{O}_s) \end{cases} \tag{4}$$

where $Int(\diamond)$ and $Ext(\diamond)$ are respectively the interior and the exterior of a subinterval $\diamond$ of $\mathbb{R}$.

2. On the other hand the functions f and μ are supposed to satisfy the following assumptions:

$$\begin{cases} \circ \quad f \in L^2(\Omega) \\ \\ \circ \quad \mu \in L^\infty(\Omega), \quad \text{with} \quad \mu(x) \geq 0 \text{ a.e. in } \Omega. \end{cases} \tag{5}$$

Based on the previous assumptions, the Lax-Milgram theorem (see for instance the well known book of H. Brezis, precisely the Corollary 5.8 at the page 140 of that book. 5.) applies and ensures existence and uniqueness of a weak solution to the problem (1)-(2), that is,

$$\begin{cases} \text{There exists a unique function } u \in H_0^1(\Omega) \text{ such that:} \\ \int_\Omega \lambda u'v'dx + \int_\Omega \mu uvdx = \int_\Omega fvdx \qquad \forall\, v \in H_0^1(\Omega) \end{cases} \tag{6}$$

where the derivatives u' and v' are understood in the sense of distributions and where $H_0^1(\Omega)$ is a well-known Sobolev space defined as

$$H_0^1(\Omega) = \{v \in L^2(\Omega);\ v' \in L^2(\Omega), \quad v(a) = v(b) = 0\}. \tag{7}$$

The space $H_0^1(\Omega)$ is endowed with its standard norm defined as

$$\| v \|_{H_0^1} = \| v' \|_{L^2(\Omega)} \qquad \forall\, v \in H_0^1(\Omega). \tag{8}$$

Since Ω is a bounded interval the previous norm is equivalent to the following one (direct consequence of Poincaré-Friedrichs inequality that we recall at the end of the current subsection):

$$\| v \| = [\| v \|_{L^2(\Omega)}^2 + \| v' \|_{L^2(\Omega)}^2]^{\frac{1}{2}} \qquad \forall\, v \in H_0^1(\Omega). \tag{9}$$

This last norm is in fact the standard norm of the well known Sobolev space $H^1(\Omega)$ we recall here:

$$H^1(\Omega) = \{v \in L^2(\Omega);\ v' \in L^2(\Omega)\}. \tag{10}$$

It is then obvious that $H_0^1(\Omega)$ is a (closed) subspace of the space $H^1(\Omega)$. From now The standard norm of $H^1(\Omega)$ is denoted by $\| \ . \ \|_{H^1(\Omega)}$ or simply $\| \ . \ \|_{H^1}$ if there is no risk of confusion. Notice finally that the previous weak formulation is the starting point for the finite element analysis of (1), (2). See for instance the following well known books by S.C. Brenner and L.R. Scott on one hand and P.G. Ciarlet on the other hand. [3, 20].

It is worth mentioning that the weak solution to the system (1)-(2) satisfies the following stability inequality:

$$\| u \|_{H_0^1(\Omega)} \leq C \| f \|_{L^2(\Omega)} \tag{11}$$

with C being a nonnegative real number. Let us end this subsection with the following important result.[5].

Proposition 1.1. (*Poincaré-Friedrichs inequality*)
Let d be a given space dimension and D a nonempty open domain of $\mathbb{R}^d$. If D is bounded in one direction at least then :

$$\forall\, v \in H_0^1(D) \qquad \| v \|_{L^2(D)} \leq C \| grad\, v \|_{L^2(\Omega)} \tag{12}$$

where C is a nonnegative real number not depending on v and $grad\, v$ the gradient of v.

Notice that in the one-dimensional space context the condition that "*D is bounded* " is essential for the previous theorem. By contrast in higher dimension spaces (2D, 3D, ...) it is only required that Ω should be bounded in one direction (at least).

2. System of Meshes for the New Finite Volume Scheme and Related Discrete Function Spaces

The new finite volume method developed in this work is based on the concept of primary relatively "coarse" mesh associated with a control-volume mesh (to be extensively defined later).

2.1. The System of Meshes Required for the New Finite Volume Scheme

The novel finite volume technique we are going to expose below requires a system made up of two classes of meshes described as it follows.

1. *Primary relatively coarse mesh*

The primary mesh is the first mesh we define over $\overline{\Omega}$ and any mesh refinement initiative is operated exclusively from that mesh, not anywhere else. Its main role is to give a precise delimitation of the different homogeneous subdomains of Ω in the context of piecewise constant diffusion coefficient. This assumption on the diffusion coefficient is very realistic for many complex engineering problems as subsurface multi-phase flow problems very studied by Petroleum Engineers: To learn more on this topic see for instance the following references and the ones therein. $[1, 2, 19]$.

Let $N \in \mathbb{N}$ be given, where $\mathbb{N}$ is the set of positive integer, and let $\left\{x_{i+\frac{1}{2}}\right\}_{i=0}^{N}$ be an increasing sequence made up of points from $\overline{\Omega} = [a; b]$, i.e.

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{i+\frac{1}{2}} < \cdots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}} = b. \tag{13}$$

Let us set what follows:

$$\Omega_i \overset{def}{=} \left[x_{i-\frac{1}{2}}; x_{i+\frac{1}{2}}\right] \qquad \forall\, i = 1, \cdots, N \tag{14}$$

$$x_i \overset{def}{=} \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2} \qquad \forall\, i = 1, \cdots, N \tag{15}$$

$$h_i \overset{def}{=} mes(\Omega_i) \qquad \forall\, i = 0, \cdots, N+1 \tag{16}$$

where $mes(.)$ stands for Lebesgue measure in the space of one dimension, with the following conventions:

$$h_0 = h_{N+1} = 0 \tag{17}$$

dictated by the fact that

$$\Omega_0 \overset{def}{=} \{x_0\}, \quad \Omega_{N+1} \overset{def}{=} \{x_{N+1}\} \tag{18}$$

where we have set:

$$x_0 = x_{\frac{1}{2}} = a \quad \text{and} \quad x_{N+1} = x_{N+\frac{1}{2}} = b. \tag{19}$$

Definition 2.1. (Primary mesh)
The family $\left(\{\Omega_i\}_{i=1}^{N}\ ;\ \{x_i\}_{i=1}^{N}\right)$ defines a primary mesh over $\overline{\Omega}$. The mesh elements Ω_i, $i = 1, \cdots, N$, are called primary mesh elements in what follows.

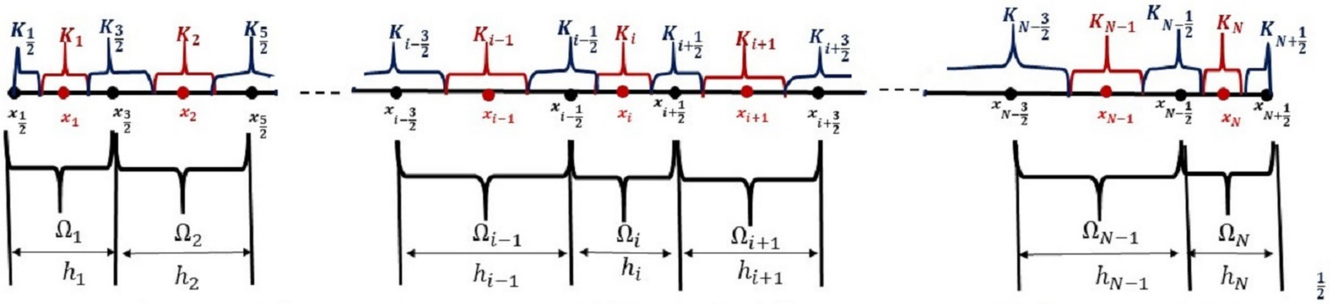

Figure 1. *Primary mesh elements in black color, control-volumes in blue and red colors.*

We introduce a parameter h defined by $h \overset{def}{=} \max_{1\leq i\leq N} h_i$ with the assumption that there exists a mesh independent nonnegative constant ρ such that

$$\frac{h}{h_i} \leq \rho \qquad \forall 1 \leq i \leq N. \tag{20}$$

As usually this parameter is called the mesh size in what follows and is assigned to tend to zero.

Important Assumption: All the discontinuity points of the diffusion coefficient λ are part of the boundaries of mesh elements $\Omega_i]_{i=1}^N$.

2. *Computation mesh or Control-volume mesh*

For computational purposes the primary mesh is associated with a finer mesh made up of two sub-families of control-volumes as indicated in Figure 1 above and in the following definition.

Definition 2.2. (*Control-volumes*)

a. *The first sub-family of control-volumes is made up of*:

$$K_i = \left[x_i - \frac{h_i}{4}\, ;\, x_i + \frac{h_i}{4}\right] \qquad i = 1, \cdots, N \tag{21}$$

with cell centers x_i, *for all* $i = 1, ..., N$.

b. *The second sub-family of control-volumes is made up of*:

$$K_{i+\frac{1}{2}} = \left[x_{i+\frac{1}{2}} - \frac{h_i}{4}\, ;\, x_{i+\frac{1}{2}} + \frac{h_{i+1}}{4}\right] \qquad i = 0, \cdots, N \tag{22}$$

with cell "centers" $x_{i+\frac{1}{2}}$, for all $i = 0, ..., N$ *and accounting with the conventions* (17).

Note that the points $x_{i+\frac{1}{2}}$, for $i \in \{0, ..., N\}$, are not necessarily the midpoints of intervals $K_{i+\frac{1}{2}}$ for $i \in \{0, ..., N\}$ (see also Figure 1).

Definition 2.3. (Computation meshes)

The sub-families $\left(\{K_i\}_{i=1}^N, \{x_i\}_{i=1}^N\right)$ *and* $\left(\left\{K_{i+\frac{1}{2}}\right\}_{i=0}^N, \left\{x_{i+\frac{1}{2}}\right\}_{i=0}^N\right)$ *define a computation mesh over* $\overline{\Omega}$, *simply denoted by* $\mathfrak{M}$.

The points $\{x_i\}_{i=1}^N$ and $\left\{x_{i+\frac{1}{2}}\right\}_{i=0}^N$ are the places where are located the discrete unknowns $\{u(x_i) \equiv u_i\}_{i=1}^N$ and $\{u(x_{i+\frac{1}{2}}) \equiv u_{i+\frac{1}{2}}\}_{i=1}^{N-1}$. The new method of finite volumes we expose below indicates a way to get second order approximations of these discrete unknowns for diffusion-reaction operators involving discontinuous diffusion coefficients. The finite volume solution to (1-2) will be denoted by $(\{\overline{u}_i\}_{i=1}^N, \{\overline{u}_{i+\frac{1}{2}}\}_{i=1}^{N-1})$.

2.2. *Main Theoretical Tools*

We present here some important theoretical tools for a finite volume analysis of the system (1-2). Among them are discrete function spaces, discrete gradient, adequate inner products and associated discrete norms, a discrete version of Poincaré-Friedrichs inequality, projection and interpolation operators. Such tools were introduced around 2005 by Pascal Omnes's team for numerical analysis of the conventional Discrete Duality Finite Volume method discovered earlier (around 2000) by Hermeline and Njifenjou - Moukouop. [8-10]

2.2.1. *Fundamental Discrete Function Spaces and Their Properties*

Let us start with introducing a basic function space denoted by $\mathbb{S}^{\mathfrak{M},0}$ (called "discrete function space" because of that it depends on the computation mesh $\mathfrak{M}$) and defined by:

$$\mathbb{S}^{\mathfrak{M},0} \overset{def}{=} \Big\{ v_h \ \ / \ v_h(x) = \sum_{i=1}^N v_i \mathbf{1}_{K_i}(x) + \sum_{j=0}^N v_{j+\frac{1}{2}} \mathbf{1}_{K_{j+\frac{1}{2}}}(x)$$
$$\text{a.e in } \Omega \text{ with } v_i, v_{j+\frac{1}{2}} \in \mathbb{R}\ \forall i = 1, \cdots, N; \forall j = 0, \cdots; N \Big\} \tag{23}$$

where we have set, for any subinterval T of Ω and for almost every $x \in \Omega$:

$$\mathbf{1}_T(x) = \begin{cases} 1 & \text{if} \quad x \in Int(T) \\ 0 & \text{if} \quad x \in Ext(T) \end{cases} \tag{24}$$

where $Int(\diamond)$ and $Ext(\diamond)$ respectively stand for interior and exterior of $\diamond$ with respect to the standard topology of $\mathbb{R}$. Let us mention the following obvious result.

Proposition 2.1. *The family of functions* $\left(\{\mathbf{1}_{K_i}\}_{i=1}^{N}, \left\{\mathbf{1}_{K_{j+\frac{1}{2}}}\right\}_{j=0}^{N}\right)$ *is a (natural) basis of the discrete function space* $\mathbb{S}^{\mathfrak{M},0}$.

So the vector space $\mathbb{S}^{\mathfrak{M},0}$ is of dimension $2N+1$. Let us introduce now an important subspace of $\mathbb{S}^{\mathfrak{M},0}$ denoted by $\mathbb{S}_0^{\mathfrak{M},0}$ and defined as :

$$\mathbb{S}_0^{\mathfrak{M},0} \stackrel{def}{=} \{v_h \in \mathbb{S}^{\mathfrak{M},0} \,/\, v_{\frac{1}{2}} = v_{N+\frac{1}{2}} = 0\}. \tag{25}$$

This subspace, with dimension $2N-1$, involves the homogeneous Dirichlet boundary conditions (2). As we will see later it plays a key role in the finite volume reconstruction of the exact solution to (1)-(2), with second order convergence rate.

Remark 2.1. *Note that when one replaces the boundary conditions* (2) *with the following one:* $u(a) = \alpha$ and $u(b) = \beta$ *the discrete function framework* $\mathbb{S}_0^{\mathfrak{M},0}$ *should be replaced with what follows:*

$$\mathbb{S}_{\alpha,\beta}^{\mathfrak{M},0} \stackrel{def}{=} \{v_h \in \mathbb{S}^{\mathfrak{M},0} \,/\, v_h(x_{\frac{1}{2}}) = \alpha \text{ and } v_h(x_{N+\frac{1}{2}}) = \beta\}. \tag{26}$$

We will come back to nonhomogeneous Dirichlet boundary conditions for further comments.

Definition 2.4. For a given bounded interval T of $\mathbb{R}$ we set:

$$\mathbb{P}_0(T) = \{v : T \longrightarrow \mathbb{R} \,/\, \exists\, v_T \in \mathbb{R} \quad s.t. \quad v(x) = v_T \quad in \quad T\}. \tag{27}$$

where "s.t." stands for such that. □

On the other hand let us set:

$$\overline{\mathbb{S}}_0^{\mathfrak{M},0} \stackrel{def}{=} \left\{ v_h \quad / \; v_h(x) = \sum_{i=1}^{N} v_i \mathbf{1}_{K_i}(x) \text{ a.e. in } \Omega \right. \\ \left. \text{with } v_i \in \mathbb{R} \;\; \forall i = 1, \cdots, N \right\} \tag{28}$$

and

$$\underline{\mathbb{S}}_0^{\mathfrak{M},0} \stackrel{def}{=} \left\{ v_h \quad / \; v_h(x) = \sum_{j=1}^{N-1} v_{j+\frac{1}{2}} \mathbf{1}_{K_{j+\frac{1}{2}}}(x) \text{ a.e in } \Omega \right. \\ \left. \text{with } v_{j+\frac{1}{2}} \in \mathbb{R} \; \forall j = 1, \cdots, N-1 \right\}, \tag{29}$$

accounting with the boundary conditions $v_{\frac{1}{2}} = v_{N+\frac{1}{2}} = 0$. Note that the function spaces $\overline{\mathbb{S}}_0^{\mathfrak{M},0}$ and $\underline{\mathbb{S}}_0^{\mathfrak{M},0}$ are obviously subspaces of $\mathbb{S}_0^{\mathfrak{M},0}$. Moreover the following properties hold:

Proposition 2.2. (*Isomorphic relations*)

We use the symbol $\equiv$ *between two vector spaces to mean there exists an isomorphism between them.*

a. *The following isomorphic relation holds:*

$$\mathbb{S}_0^{\mathfrak{M},0} \stackrel{def}{=} \overline{\mathbb{S}}_0^{\mathfrak{M},0} \oplus \underline{\mathbb{S}}_0^{\mathfrak{M},0} \quad \equiv \quad \overline{\mathbb{S}}_0^{\mathfrak{M},0} \times \underline{\mathbb{S}}_0^{\mathfrak{M},0}.$$

b. *Moreover we have the following isomorphic relations:*

$$\overline{\mathbb{S}}_0^{\mathfrak{M},0} \equiv \prod_{i=1}^{N} \mathbb{P}_0(K_i) \quad \text{and} \quad \underline{\mathbb{S}}_0^{\mathfrak{M},0} \equiv \prod_{i=1}^{N-1} \mathbb{P}_0(K_{i+\frac{1}{2}}).$$

2.2.2. Discrete Gradient Operator

The definition of discrete gradient operator requires the introduction of the mesh $\mathfrak{D}$ made up of elements from one of the forms: $D_{i+\frac{1}{4}} = [x_i, x_{i+\frac{1}{2}}]$ or $D_{i-\frac{1}{4}} = [x_{i-\frac{1}{2}}, x_i]$, for $i \in \{1, 2, ..., N\}$.

So we have

$$\mathfrak{D} =\Big\{D_{1-\frac{1}{4}}, D_{1+\frac{1}{4}}, D_{2-\frac{1}{4}}, D_{2+\frac{1}{4}}, \dots, D_{i-\frac{1}{4}}, D_{i+\frac{1}{4}}, \dots, D_{N-\frac{1}{4}}, D_{N+\frac{1}{4}}\Big\}.$$

Another ingredient is the discrete function space $\mathbb{S}^{\mathfrak{D},0}$ defined as

$$\mathbb{S}^{\mathfrak{D},0} = \{\zeta_h \,/\, \exists\{(\zeta_{i-\frac{1}{4}}, \zeta_{i+\frac{1}{4}})\}_{i=1}^{N} \subset \mathbb{R}^2 \quad \text{such that} \quad \zeta_h(x) = $$
$$= \sum_{i=1}^{N} \Big[\zeta_{i-\frac{1}{4}} \mathbf{1}_{D_{i-\frac{1}{4}}}(x) \,+\, \zeta_{i+\frac{1}{4}} \mathbf{1}_{D_{i+\frac{1}{4}}}(x)\Big] \quad a.e.\ in\ \Omega\} \tag{30}$$

We equip the space $\mathbb{S}^{\mathfrak{D},0}$ with the following scalar product:

Definition 2.5. (*Scalar Product*)

We define over $\mathbb{S}^{\mathfrak{D},0}$ a scalar product $(.,.)_{L^2(\Omega),\mathfrak{D}}$ as follows :

$$(\zeta_h, \xi_h)_{L^2(\Omega),\mathfrak{D}} = \sum_{i=1}^{N} \frac{h_i}{2} \Big[\zeta_{i-\frac{1}{4}}\xi_{i-\frac{1}{4}} + \zeta_{i+\frac{1}{4}}\xi_{i+\frac{1}{4}}\Big] \quad \forall \zeta_h, \xi_h \in \mathbb{S}^{\mathfrak{D},0}.$$

The previous scalar product is associated with the following norm :

$$\|\zeta_h\|_{L^2(\Omega),\mathfrak{D}} \;=\; \left(\sum_{i=1}^{N} \frac{h_i}{2}\Big[\zeta^2_{i+\frac{1}{4}} \;+\; \zeta^2_{i-\frac{1}{4}}\Big]\right)^{\frac{1}{2}} \quad \forall\, \zeta_h \in \mathbb{S}^{\mathfrak{D},0}. \tag{31}$$

Definition 2.6. (*Discrete Gradient operator*)

A linear operator $\nabla^{\mathfrak{D}}$ from $\mathbb{S}^{\mathfrak{M},0}$ to $\mathbb{S}^{\mathfrak{D},0}$ is called a discrete gradient operator if and only if for all $v_h \in \mathbb{S}^{\mathfrak{M},0}$ the following identity holds :

$$\nabla^{\mathfrak{D}} v_h \;\stackrel{def}{=}\; \sum_{i=1}^{N} \frac{2}{h_i}\Big[(v_i - v_{i-\frac{1}{2}})\mathbf{1}_{D_{i-\frac{1}{4}}} \;+\; (v_{i+\frac{1}{2}} - v_i)\mathbf{1}_{D_{i+\frac{1}{4}}}\Big] \tag{32}$$

Setting:

$$[\nabla^{\mathfrak{D}} v_h]_{i+\frac{1}{4}} = \frac{v_{i+\frac{1}{2}} - v_i}{h_i/2} \quad \text{and} \quad [\nabla^{\mathfrak{D}} v_h]_{i-\frac{1}{4}} = \frac{v_i - v_{i-\frac{1}{2}}}{h_i/2} \tag{33}$$

the following obvious result holds:

Proposition 2.3. *The mapping*

$$v \longmapsto \left\|\nabla^{\mathfrak{D}} v_h\right\|_{L^2(\Omega),\mathfrak{D}} \stackrel{def}{=} \left(\sum_{i=1}^{N} \frac{h_i}{2}\Big[\big[\nabla^{\mathfrak{D}} v_h\big]^2_{i+\frac{1}{4}} + \big[\nabla^{\mathfrak{D}} v_h\big]^2_{i-\frac{1}{4}}\Big]\right)^{\frac{1}{2}} \quad \forall v_h \in \mathbb{S}^{\mathfrak{M},0}$$

is a semi-norm over the discrete function space $\mathbb{S}^{\mathfrak{M},0}$.

Remark 2.2. *It is to notice that the mesh $\mathfrak{D}$ could be defined as the family $\{D_{z_j}\}_{j=1}^{2N}$, where we have set for $j \in \{1, 2, 3, ..., 2N\}$:*

$$z_j \;=\; \frac{3}{4} + \frac{1}{2}(j-1) \quad \text{and} \quad D_{z_j} \;=\; [x_{z_j - \frac{1}{4}}, x_{z_j + \frac{1}{4}}].$$

It then follows from what precedes that the discrete gradient $\nabla^{\mathfrak{D}}$ of a discrete function v_h from the space $\mathbb{S}^{\mathfrak{M},0}$ could be also defined as:

$$\nabla^{\mathfrak{D}} v_h \;=\; \sum_{j=1}^{2N} \frac{1}{x_{z_j+\frac{1}{4}} - x_{z_j-\frac{1}{4}}}\Big[v_{z_j+\frac{1}{4}} - v_{z_j-\frac{1}{4}}\Big]\,\mathbf{1}_{D_{z_j}}.$$

2.2.3. Inner Products Defined on $\mathbb{S}^{\mathfrak{M},0}$ and $\mathbb{S}_0^{\mathfrak{M},0}$

Recall that $\mathbb{S}_0^{\mathfrak{M},0}$ is a subspace of $\mathbb{S}^{\mathfrak{M},0}$. Let us define over the space $\mathbb{S}^{\mathfrak{M},0}$ the following inner products (where $mes[.]$ is the Lebesgue measure in one-dimensional space):

1. *First inner product on $\mathbb{S}^{\mathfrak{M},0}$ and particular case of $\mathbb{S}_0^{\mathfrak{M},0}$*

$$(v_h, w_h)_{L^2(\Omega),\mathfrak{M}} \overset{def}{=} \sum_{i=1}^{N} mes[K_i]v_i w_i + \sum_{i=0}^{N} mes[K_{i+\frac{1}{2}}]v_{i+\frac{1}{2}} w_{i+\frac{1}{2}} \ \forall v_h, w_h \in \mathbb{S}^{\mathfrak{M},0} \quad (34)$$

with associated norm $\| \,.\, \|_{L^2(\Omega),\mathfrak{M}}$ naturally defined by:

$$\| v_h \|_{L^2(\Omega),\mathfrak{M}} \overset{def}{=} = \left(\sum_{i=1}^{N} \sum_{i=1}^{N} mes[K_i]v_i^2 + \sum_{i=0}^{N} mes[K_{i+\frac{1}{2}}]v_{i+\frac{1}{2}}^2 \right)^{\frac{1}{2}} \quad \forall v_h \in \mathbb{S}^{\mathfrak{M},0}. \quad (35)$$

The restriction of the norm $\|\|_{L^2(\Omega),\mathfrak{M}}$ to the space $\mathbb{S}_0^{\mathfrak{M},0}$ reads as follows:

$$\| v_h \|^*_{L^2(\Omega),\mathfrak{M}} \overset{def}{=} = \left(\sum_{i=1}^{N} \sum_{i=1}^{N} mes[K_i]v_i^2 + \sum_{i=1}^{N-1} mes[K_{i+\frac{1}{2}}]v_{i+\frac{1}{2}}^2 \right)^{\frac{1}{2}} \quad \forall v_h \in \mathbb{S}_0^{\mathfrak{M},0} \quad (36)$$

since $v_{\frac{1}{2}} = v_{N+\frac{1}{2}} = 0$ for all $v_h \in \mathbb{S}_0^{\mathfrak{M},0}$ (see relation (25)).

2. *Second inner product on* $\mathbb{S}^{\mathfrak{M},0}$ *:*

$$(v_h, w_h)_{\mathcal{H}^1(\Omega),\mathfrak{M},\mathfrak{D}} \overset{def}{=} (v_h, w_h)_{L^2(\Omega),\mathfrak{M}} + \left(\nabla^{\mathfrak{D}} v_h, \nabla^{\mathfrak{D}} w_h\right)_{L^2(\Omega),\mathfrak{D}} \ \forall v_h, w_h \in \mathbb{S}^{\mathfrak{M},0}. \quad (37)$$

The discrete norm associated with this inner product is denoted by $\|v_h\|_{\mathcal{H}^1(\Omega),\mathfrak{M},\mathfrak{D}}$ and naturally defined by

$$\|v_h\|_{\mathcal{H}^1(\Omega),\mathfrak{M},\mathfrak{D}} = \left(\|v_h\|^2_{L^2(\Omega),\mathfrak{M}} + \left\|\nabla^{\mathfrak{D}} v_h\right\|^2_{L^2(\Omega),\mathfrak{D}} \right)^{\frac{1}{2}} \quad \forall v_h \in \mathbb{S}_0^{\mathfrak{M},0}. \quad (38)$$

3. *Discrete version of Poincaré-Friedrichs inequality :*

We recall that the "continuous" version of Poincaré-Friedrichs inequality is given by (12). Let us start with introducing some projection operators at least useful for the proof of the discrete version of Poincaré-Friedrichs inequality.

Definition 2.7. (*Projection operators*)

Let $\overline{\mathbb{P}}_{\mathfrak{M}}$ *and* $\underline{\mathbb{P}}_{\mathfrak{M}}$ *be two projection operators defined on the space* $\mathbb{S}_0^{\mathfrak{M},0}$ *with values in* $\overline{\mathbb{S}}_0^{\mathfrak{M},0}$ *and* $\underline{\mathbb{S}}_0^{\mathfrak{M},0}$ *respectively* (*see relations* (25) (28) *and* (29) *for the definition of these spaces*):

$$\overline{\mathbb{P}}_{\mathfrak{M}} : \ \mathbb{S}_0^{\mathfrak{M},0} \quad \longrightarrow \quad \overline{\mathbb{S}}_0^{\mathfrak{M},0} \quad \textit{such that} \quad \overline{\mathbb{P}}_{\mathfrak{M}}(v_h) = \sum_{i=1}^{N} v_i \mathbf{1}_{K_i} \quad (39)$$

$$\textit{and } \underline{\mathbb{P}}_{\mathfrak{M}} : \ \mathbb{S}_0^{\mathfrak{M},0} \quad \longrightarrow \quad \underline{\mathbb{S}}_0^{\mathfrak{M},0} \quad \textit{such that} \quad \underline{\mathbb{P}}_{\mathfrak{M}}(v_h) = \sum_{i=1}^{N-1} v_{i+\frac{1}{2}} \mathbf{1}_{K_{i+\frac{1}{2}}}. \quad (40)$$

Lemma 2.1. *The projection operators* $\overline{\mathbb{P}}_{\mathfrak{M}}$ *and* $\underline{\mathbb{P}}_{\mathfrak{M}}$ *satisfy the following continuity properties:* $\forall\, v_h \in \mathbb{S}_0^{\mathfrak{M},0}$

$$\begin{cases} \|\overline{\mathbb{P}}_{\mathfrak{M}}(v_h)\|_{L^2(\Omega),\mathfrak{M}} \ \leq \ mes[\Omega] \|\nabla^{\mathfrak{D}} v_h\|_{L^2(\Omega),\mathfrak{D}} \\ \|\underline{\mathbb{P}}_{\mathfrak{M}}(v_h)\|_{L^2(\Omega),\mathfrak{M}} \ \leq \ mes[\Omega] \|\nabla^{\mathfrak{D}} v_h\|_{L^2(\Omega),\mathfrak{D}} \end{cases} \quad (41)$$

Proof a. Let us prove the first inequality given by the previous Lemma i.e $\forall\, v_h \in \mathbb{S}_0^{\mathfrak{M},0}$

$$\|\overline{\mathbb{P}}_{\mathfrak{M}}(v_h)\|_{L^2(\Omega),\mathfrak{M}} \ \leq \ mes[\Omega] \|\nabla^{\mathfrak{D}} v_h\|_{L^2(\Omega),\mathfrak{D}}.$$

Let us choose arbitrarily $i \in \{1, 2, ..., N\}$ and $v_h \in \mathbb{S}_0^{\mathfrak{M},0}$. So for a.e. $x \in K_i$ we have what follows:

$$v_h(x) \equiv v_i = [v_1 - v_{1-\frac{1}{2}}] + [-v_1 + v_{1+\frac{1}{2}}] + [v_2 - v_{2-\frac{1}{2}}] + [-v_2 + v_{2-\frac{1}{2}}] + \ldots + [v_i - v_{i-\frac{1}{2}}]. \quad (42)$$

We deduce that for a.e. $x \in K_i$ the following holds:

$$\begin{aligned} \mid v_h(x) \mid = \mid v_i \mid &\leq \mid v_1 - v_{1-\frac{1}{2}} \mid + \mid -v_1 + v_{1+\frac{1}{2}} \mid + \mid v_2 - v_{2-\frac{1}{2}} \mid + \mid -v_2 + v_{2-\frac{1}{2}} \mid + \ldots + \mid v_i - v_{i-\frac{1}{2}} \mid \\ &\leq \sum_{j=1}^{N} \Big(\mid v_j - v_{j-\frac{1}{2}} \mid + \mid v_j - v_{j+\frac{1}{2}} \mid \Big) \\ &\leq \sum_{j=1}^{N} \left[\sqrt{h_j/2} \Big(\mid v_j - v_{j-\frac{1}{2}} \mid + \mid v_j - v_{j+\frac{1}{2}} \mid \Big) \frac{1}{\sqrt{h_j/2}} \right] \end{aligned} \tag{43}$$

It follows from Cauchy-Schawrz inequality that for a.e. $x \in K_i$, with $i \in \{1, 2, ..., N\}$, we have:

$$\mid v_h(x) \mid^2 = \mid v_i^2 \mid \leq mes[\Omega] \sum_{j=1}^{N} \frac{2}{h_j} \left[\Big(v_j - v_{j-\frac{1}{2}} \Big)^2 + \Big(v_j - v_{j+\frac{1}{2}} \Big)^2 \right] = mes[\Omega] \parallel \nabla^{\mathfrak{D}} v_h \parallel^2_{L^2(\Omega),\mathfrak{D}} \tag{44}$$

Integrating the both sides of the previous inequality in K_i and summing on $i \in \{1, 2, ..., N\}$ leads straightly to the investigated inequality.

b. Let us prove the second inequality given by the previous Lemma i.e.

$$\forall\, v_h \in \mathbb{S}_0^{\mathfrak{M},0} \, \|\mathbb{P}_{\mathfrak{M}}(v_h)\|_{L^2(\Omega),\mathfrak{M}} \;\leq\; mes[\Omega] \|\nabla^{\mathfrak{D}} v_h\|_{L^2(\Omega),\mathfrak{D}}.$$

As for the first inequality, let us choose arbitrarily $i \in \{1, 2, ..., N\}$ and $v_h \in \mathbb{S}_0^{\mathfrak{M},0}$. So for a.e. $x \in K_{i+\frac{1}{2}}$ we have what follows:

$$v_h(x) = v_{i+\frac{1}{2}} = (v_{i+\frac{1}{2}} - v_i) + (v_i - v_{i-\frac{1}{2}}) + (v_{i-\frac{1}{2}} - v_{i-1}) + (v_{i-1} - v_{i-\frac{3}{2}}) + (v_{i-\frac{3}{2}} - v_{i-2}) + \ldots\ldots + (v_i - v_{\frac{1}{2}}).$$

It follows that for a.e. $x \in K_{i+\frac{1}{2}}$, with $i \in \{1, 2, ..., N-1\}$, we have

$$\mid v_h(x) \mid = \mid v_{i+\frac{1}{2}} \mid \leq \sum_{j=1}^{N} \left[\sqrt{h_j/2} \Big(\mid v_j - v_{j+\frac{1}{2}} \mid + \mid v_j - v_{j-\frac{1}{2}} \mid \Big) \frac{1}{\sqrt{h_j/2}} \right]$$

Thanks to Cauchy-Schwarz we get for a.e. $x \in K_{i+\frac{1}{2}}$, with $i \in \{1, 2, ..., N-1\}$,

$$\mid v_h(x) \mid^2 = \mid v_{i+\frac{1}{2}} \mid^2 \leq mes[\Omega] \|\nabla^{\mathfrak{D}} v_h\|^2_{L^2(\Omega),\mathfrak{D}} \tag{45}$$

Integrating the both sides of the previous inequality in $K_{i+\frac{1}{2}}$ and summing on $i \in \{1, 2, ..., N-1\}$ lead to the second inequality of the previous Lemma.

We have the following results that serve as ingredients for the stability result and error estimates coming in the last section. Let us start with setting.

$$\parallel v_h \parallel_{L^\infty(\Omega),\mathfrak{M}} = max\{\mid v_1 \mid, \mid v_{\frac{3}{2}} \mid, \mid v_2 \mid, \mid v_{\frac{3}{2}} \mid, \ldots, \ldots, \mid v_{N-\frac{1}{2}} \mid, \mid v_N \mid]\} \quad \forall v_h \in \mathbb{S}_0^{\mathfrak{M},o}$$

Proposition 2.4. (*Discrete Poincaré-Friedrichs inequality and plus*)

c. *The following discrete version of Poincaré-Friedrichs inequality holds:*

$$\parallel v_h \parallel_{L^2(\Omega),\mathfrak{M}} \leq \sqrt{2} mes[\Omega] \|\nabla^{\mathfrak{D}} v_h\|_{L^2(\Omega),\mathfrak{D}} \quad \forall\, v_h \in \mathbb{S}_0^{\mathfrak{M},0}. \tag{46}$$

d. *Moreover we have:*

$$\parallel v_h \parallel_{L^\infty(\Omega),\mathfrak{M}} \leq \sqrt{mes[\Omega]} \|\nabla^{\mathfrak{D}} v_h\|_{L^2(\Omega),\mathfrak{D}} \quad \forall\, v_h \in \mathbb{S}_0^{\mathfrak{M},0}. \tag{47}$$

Proof Summing side by side the two inequalities of the previous Lemma leads straightly to the discrete version of Poincaré-Friedrichs inequality as stated in the previous Proposition. The second inequality follows straightly from (44)-(45).

4. *Standard inner product of* $\mathbb{S}_0^{\mathfrak{M},0}$

An immediate consequence of the previous Proposition is that:

Corollary 2.1. (*Very useful result*)

The following mapping defined over $\mathbb{S}^{\mathfrak{M},0}$ *by*

$$v_h \quad \longmapsto \quad \|\nabla^{\mathfrak{D}} v_h\|_{L^2(\Omega),\mathfrak{D}} \tag{48}$$

is a norm for $\mathbb{S}_0^{\mathfrak{M},0}$*. Moreover this norm is equivalent on* $\mathbb{S}_0^{\mathfrak{M},0}$ *to the standard norm of* $\mathbb{S}^{\mathfrak{M},0}$ *introduced above and associated with the scalar product* $(.,.)_{\mathcal{H}^1(\Omega),\mathfrak{M},\mathfrak{D}}$ (see relation (37)).

Terminology: The norm defined on $\mathbb{S}_0^{\mathfrak{M},0}$ by (48) is called in the sequel "Discrete $\mathcal{H}_0^1-$norm" and considered as the standard norm of this discrete functional space.

$$\begin{aligned}(v_h, w_h)_{\mathcal{H}^1(\Omega),\mathfrak{M},\mathfrak{D}} &\stackrel{def}{=} (v_h, w_h)_{L^2(\Omega),\mathfrak{M}} + \\ + \left(\nabla^{\mathfrak{D}} v_h, \nabla^{\mathfrak{D}} w_h\right)_{L^2(\Omega),\mathfrak{D}} & \qquad \forall v_h, w_h \in \mathbb{S}^{\mathfrak{M},0}\end{aligned} \tag{49}$$

defines a scalar product on $\mathbb{S}_0^{\mathfrak{M},0}$, called in the sequel "standard scalar product" of $\mathbb{S}_0^{\mathfrak{M},0}$. This scalar product is associated with the "Discrete $\mathcal{H}_0^1-$norm".

Let us give now a result that plays a key role in the proof of the stability of the new finite volume scheme that we will be exposing in the next section.

Proposition 2.5. (*A fundamental result*)

Let $\Pi_{\mathfrak{M}}$ *be a linear mapping defined over* $L^2(\Omega)$ *with values in the space* $\mathbb{S}^{\mathfrak{M},0} \stackrel{def}{=} \overline{\mathbb{S}}_0^{\mathfrak{M},0} \oplus \underline{\mathbb{S}}^{\mathfrak{M},0}$ *as follows:*

$$\Pi_{\mathfrak{M}}(v) = \sum_{i=1}^{N} \langle v\rangle_i \mathbf{1}_{K_i} + \sum_{i=0}^{N} \langle v\rangle_{i+\frac{1}{2}} \mathbf{1}_{K_{i+\frac{1}{2}}} \qquad \forall\, v \in L^2(\Omega) \tag{50}$$

where $\underline{\mathbb{S}}^{\mathfrak{M},0} \stackrel{def}{=} \underline{\mathbb{S}}_0^{\mathfrak{M},0} \oplus Vect(\mathbf{1}_{K_{\frac{1}{2}}}, \mathbf{1}_{K_{N+\frac{1}{2}}})$ *and where we have set:*

$$\langle v\rangle_i = \frac{1}{mes(K_i)} \int_{K_i} v(x)dx \qquad \forall\, 1 \le i \le N \tag{51}$$

and

$$\langle v\rangle_{i+\frac{1}{2}} = \frac{1}{mes(K_{i+\frac{1}{2}})} \int_{K_{i+\frac{1}{2}}} v(x)dx \qquad \forall\, 0 \le i \le N. \tag{52}$$

Then $\Pi_{\mathfrak{M}}$ *is continuous with respect to the norms* $\|.\|_{L^2(\Omega)}$ and $\|.\|_{L^2(\Omega),\mathfrak{M}}$ *respectively defined in* $L^2(\Omega)$ *and* $\mathbb{S}^{\mathfrak{M},0}$, i.e. *there exists* ϖ*, a mesh-independent nonnegative real number, such that*

$$\|\Pi_{\mathfrak{M}}(v)\|_{L^2(\Omega),\mathfrak{M}} \le \varpi \|v\|_{L^2(\Omega)} \qquad \forall\, v \in L^2(\Omega).$$

Proof Let v be a function from the space $L^2(\Omega)$. So according to the definition of the operator $\Pi_{\mathfrak{M}}$ (see relations (50)-(52)) we have:

$$\|\Pi_{\mathfrak{M}}(v)\|^2_{L^2(\Omega),\mathfrak{M}} = \sum_{i=1}^{N} h_i \langle v\rangle_i^2 + \sum_{i=0}^{N} h_{i+\frac{1}{2}} \langle v\rangle^2_{i+\frac{1}{2}} = \tag{53}$$

$$= \sum_{i=1}^{N} \frac{1}{h_i} [\int_{K_i} v(x)dx]^2 + \sum_{i=0}^{N} \frac{1}{h_{i+\frac{1}{2}}} [\int_{K_{i+\frac{1}{2}}} v(x)dx]^2 \le \tag{54}$$

$$\le \sum_{i=1}^{N} \int_{K_i} |v(x)|^2 dx + \sum_{i=0}^{N} \int_{K_{i+\frac{1}{2}}} |v(x)|^2 dx = \tag{55}$$

$$= \|v\|^2_{L^2(\Omega)} \tag{56}$$

According to what precedes we have established that

$$\|\Pi_{\mathfrak{M}}(v)\|_{L^2(\Omega),\mathfrak{M}} \le \|v\|_{L^2(\Omega)}. \tag{57}$$

So ends the proof.

3. A New Finite Volume Scheme for the System (1)-(2)

We expose here the main steps leading to a new finite volume approximation of the balance equation (1), with prescribed homogeneous Dirichlet boundary conditions (2).

3.1. Notations and Definitions

Let $\varphi(.)$ be a (real-valued or vector-valued) function defined over Ω and $x \in \overline{\Omega}$. We denote by $\varphi(x^+)$ and $\varphi(x^-)$ respectively the limits of $\varphi(s)$ when $s \overset{>}{\longrightarrow} x$ and when $s \overset{<}{\longrightarrow} x$. In the same order of ideas if $\zeta(.,.)$ is a (real-valued or vector-valued) function defined over $\Omega \times \mathbb{R}$, we denote by $\zeta(x^+, z)$ and $\zeta(x^-, z)$ respectively the limits of $\zeta(s, z)$ when $s \overset{>}{\longrightarrow} x$ and when $s \overset{<}{\longrightarrow} x$ for all $z \in \mathbb{R}$.

Definition 3.1. (*Flow Velocity and Flux*)

Recall that u is the exact solution to (1)-(2) *and u' is its gradient* (*in one dimension space*). *Let us define the flow velocity q and the corresponding flux F at the right-hand side and the left-hand side of every point $x \in \overline{\Omega}$ as follows*:

$$\begin{cases} q(x^\tau, u') \overset{def}{=} -[\lambda\, u'](x^\tau) \\ \\ F(x^\tau, u') \overset{def}{=} q(x^\tau, u') \cdot \nu(x^\tau) \end{cases} \tag{58}$$

where we have set

$$\nu(x^\tau) \overset{def}{=} \begin{cases} -1 & \textit{if} \quad \tau = + \\ \\ +1 & \textit{if} \quad \tau = - \end{cases} \tag{59}$$

3.2. Discrete Mass Conservation Per Control-volume

We proceed in several steps as it follows.

1. *Step one: Mass conservation principle for control-volumes*

Integrating the balance equation (1) in the control-volumes K_i and $K_{i+\frac{1}{2}}$, leads to

$$\begin{cases} F(x^-_{i+\frac{1}{4}}, u') + F(x^+_{i-\frac{1}{4}}, u') + \int_{K_i} \mu(x) u(x) dx = \frac{h_i}{2} \langle f \rangle_i \quad \forall i = 1, \ldots, N \\ F(x^-_{i+\frac{3}{4}}, u') + F(x^+_{i+\frac{1}{4}}, u') + \int_{K_{i+\frac{1}{2}}} \mu(x) u(x) dx = \frac{h_i + h_{i+1}}{4} \langle f \rangle_{i+\frac{1}{2}} \quad \forall i = 1, \ldots, N-1 \end{cases} \tag{60}$$

where we have set

$$\begin{cases} \langle f \rangle_i \overset{def}{=} \frac{1}{mes(K_i)} \int_{K_i} f(x) dx \\ \\ \langle f \rangle_{i+\frac{1}{2}} \overset{def}{=} \frac{1}{mes(K_{i+\frac{1}{2}})} \int_{K_{i+\frac{1}{2}}} f(x) dx \end{cases} \tag{61}$$

and where

$$x_{i \pm \frac{1}{4}} \overset{def}{=} x_i \pm \frac{h_i}{4}, \qquad i = 1, 2, ..., N. \tag{62}$$

Remark that the term

$$x_{i+\frac{3}{4}} \overset{def}{=} x_{i+\frac{1}{2}} + \frac{h_{i+1}}{4} \tag{63}$$

appearing in the second equation of the system (60) is of the form (62). Indeed we have

$$x_{i+\frac{3}{4}} = x_{(i+1)-\frac{1}{4}} \qquad i = 0, 1, ..., N-1. \tag{64}$$

It follows from definitions (58) that

$$\forall\forall i = 1, \ldots, N \quad \begin{cases} F(x^-_{i+\frac{1}{4}}, u') = -\lambda_i u'(x_i + \frac{h_i}{4}) \\ \\ F(x^+_{i-\frac{1}{4}}, u') = \lambda_i u'(x_i - \frac{h_i}{4}) \end{cases} \tag{65}$$

and

$$\forall\forall i = 1, \ldots, N \quad \begin{cases} F(x^-_{i+\frac{3}{4}}, u') = -\lambda_{i+1} u'(x_{i+\frac{1}{2}} + \frac{h_{i+1}}{4}) F(x^+_{i+\frac{1}{4}}, u') = \lambda_i u'(x_{i+\frac{1}{2}} - \frac{h_i}{4}). \\ \\ \end{cases} \tag{66}$$

2. *Step two: Definition of discrete flux function*

Let us investigate now second order approximations of the following flux terms: $F(x^-_{i+\frac{1}{4}}, u'), F(x^+_{i-\frac{1}{4}}, u'), F(x^-_{i+\frac{3}{4}}, u')$ and

$F(x^+_{i+\frac{1}{4}}, u')$. Under the assumption that the exact solution u to the continuous problem (1)-(2) lies in $C^3(\overline{\Omega}_i)$, for all $i \in \{1, 2, ..., N\}$, the Taylor-Lagrange theorem ensures that there exist $z^d_{i+\frac{1}{4}}$ and $z^g_{i+\frac{1}{4}}$ respectively in the intervals $]x_{i+\frac{1}{4}}, x_{i+\frac{1}{2}}[$ and $]x_i, x_{i+\frac{1}{4}}[$ such that

$$u(x_{i+\frac{1}{2}}) = u(x_i + \frac{h_i}{4}) + \frac{h_i}{4}u'(x_i + \frac{h_i}{4}) + \frac{h_i^2}{32}u''(x_i + \frac{h_i}{4}) + (\frac{h_i}{4})^3 u'''(z^d_{i+\frac{1}{4}}) \tag{67}$$

$$u(x_i) = u(x_i + \frac{h_i}{4}) - \frac{h_i}{4}u'(x_i + \frac{h_i}{4}) + \frac{h_i^2}{32}u''(x_i + \frac{h_i}{4}) - (\frac{h_i}{4})^3 u'''(z^g_{i+\frac{1}{4}}). \tag{68}$$

Subtracting (68) from (67) side by side leads to

$$u'(x_i + \frac{h_i}{4}) \;=\; \frac{\Big[u(x_{i+\frac{1}{2}}) - u(x_i)\Big]}{h_i/2} + \frac{h_i^2}{32}u'''(z_{i+\frac{1}{4}}) \quad \forall i = 1, \cdots, N. \tag{69}$$

where $x_i < z^g_{i+\frac{1}{4}} < z_{i+\frac{1}{4}} < z^d_{i+\frac{1}{4}} < x_{i+\frac{1}{2}}$. A similar development yields what follows:

$$u'(x_i - \frac{h_i}{4}) = \frac{\Big[u(x_i) - u(x_{i-\frac{1}{2}})\Big]}{h_i/2} + \frac{h_i^2}{32}u'''(r_{i-\frac{1}{4}}), \qquad \forall i = 1, \cdots, N. \tag{70}$$

with $x_{i-\frac{1}{2}} < r_{i+\frac{1}{4}} < x_i$. Therefore, thanks to (65), we can deduce that $\forall i = 1, \cdots, N$

$$\begin{cases} F(x^-_{i+\frac{1}{4}}, u') \;=\; \overline{F}(x^-_{i+\frac{1}{4}}, \nabla^{\mathfrak{D}} u_h) - R^-_{i+\frac{1}{4}}(h, u''')F(x^+_{i-\frac{1}{4}}, u') \;=\; \overline{F}(x^+_{i-\frac{1}{4}}, \nabla^{\mathfrak{D}} u_h) - R^+_{i-\frac{1}{4}}(h, u''') \end{cases} \tag{71}$$

where, according to Definition 2.6, we have

$$\nabla^{\mathfrak{D}} u_h \overset{def}{=} \sum_{i=1}^{N} \frac{1}{h_i/2}\Bigg[(u_i - u_{i-\frac{1}{2}})\mathbf{1}_{D_{i-\frac{1}{4}}} + (u_{i+\frac{1}{2}} - u_i)\mathbf{1}_{D_{i+\frac{1}{4}}} + \Bigg]$$

and where we have set

$$\forall i = 1, \cdots, N \qquad \begin{cases} \overline{F}(x^-_{i+\frac{1}{4}}, \nabla^{\mathfrak{D}} u_h) = -\lambda_i[\nabla^{\mathfrak{D}} u_h]_{i+\frac{1}{4}} \\ \overline{F}(x^+_{i-\frac{1}{4}}, \nabla^{\mathfrak{D}} u_h) = \lambda_i[\nabla^{\mathfrak{D}} u_h]_{i-\frac{1}{4}}, \end{cases} \tag{72}$$

with $[\nabla^{\mathfrak{D}} u_h]_{i+\frac{1}{4}}$ and $[\nabla^{\mathfrak{D}} u_h]_{i-\frac{1}{4}}$ defined as it follows:

Definition 3.2.

$$\forall i = 1, \cdots, N \qquad \begin{cases} [\nabla^{\mathfrak{D}} u_h]_{i+\frac{1}{4}} \overset{def}{=} \dfrac{\Big[u(x_{i+\frac{1}{2}}) - u(x_i)\Big]}{h_i/2} \\[2ex] [\nabla^{\mathfrak{D}} u_h]_{i-\frac{1}{4}} \overset{def}{=} \dfrac{\Big[u(x_i) - u(x_{i-\frac{1}{2}})\Big]}{h_i/2}. \end{cases} \tag{73}$$

Using the same arguments as for the computation of the fluxes $F(x^-_{i+\frac{1}{4}}, u')$ and $F(x^+_{i-\frac{1}{4}}, u')$ (see (71) above) leads for all $i = 1, \cdots, N-1$ to

$$\begin{cases} u'(x_{i+\frac{1}{2}} + \frac{h_{i+1}}{4}) = \frac{\Big[u(x_{i+1}) - u(x_{i+\frac{1}{2}})\Big]}{h_{i+1}/2} + \frac{h_{i+1}^2}{32}u'''(\widehat{z}_{i+\frac{3}{4}}) \\[2ex] u'(x_{i+\frac{1}{2}} - \frac{h_i}{4}) = \frac{\Big[u(x_{i+\frac{1}{2}}) - u(x_i)\Big]}{h_i/2} + \frac{h_i^2}{32}u'''(\widehat{r}_{i+\frac{1}{4}}) \end{cases} \tag{74}$$

Therefore, thanks to (66), we deduce that for all $i = 1, \cdots, N-1$

$$\begin{cases} F(x^-_{i+\frac{3}{4}}, u') \;=\; \overline{F}(x^-_{i+\frac{3}{4}}, \nabla^{\mathfrak{D}} u_h) \;-\; R^-_{i+\frac{3}{4}}(h, u''') \\ \\ F(x^+_{i+\frac{1}{4}}, u') \;=\; \overline{F}(x^+_{i+\frac{1}{4}}, \nabla^{\mathfrak{D}} u_h) \;-\; R^+_{i+\frac{1}{4}}(h, u''') \end{cases} \tag{75}$$

where $\overline{F}(x^-_{i+\frac{3}{4}}, \nabla^{\mathfrak{D}} u_h)$ and $\overline{F}(x^+_{i+\frac{1}{4}}, \nabla^{\mathfrak{D}} u_h)$ are defined as it follows:

Definition 3.3. $\forall i = 1, \cdots, N-1$

$$\begin{cases} \overline{F}(x^-_{i+\frac{3}{4}}, \nabla^{\mathfrak{D}} u_h) \overset{def}{=} -\lambda_{i+1}[\nabla^{\mathfrak{D}} u_h]_{i+\frac{3}{4}} \\ \overline{F}(x^+_{i+\frac{1}{4}}, \nabla^{\mathfrak{D}} u_h) \overset{def}{=} \lambda_i[\nabla^{\mathfrak{D}} u_h]_{i+\frac{1}{4}} \end{cases} \tag{76}$$

where $[\nabla^{\mathfrak{D}} u_h]_{i+\frac{1}{4}}$ is defined by (73) and $[\nabla^{\mathfrak{D}} u_h]_{i+\frac{3}{4}}$ is defined as it follows:

$$\forall i = 1, \cdots, N-1 \qquad [\nabla^{\mathfrak{D}} u_h]_{i+\frac{3}{4}} \overset{def}{=} \frac{\Big[u(x_{i+1}) \;-\; u(x_{i+\frac{1}{2}})\Big]}{h_{i+1}/2}. \tag{77}$$

Recall that $\{\mathcal{O}_s\}_{s=1}^S$ is a family of open (nonempty) intervals defining a partition of Ω and associated with the diffusion coefficient λ in the sense that:

$\lambda(x) = \sum_{s=1}^S \lambda_s \mathbf{1}_{\mathcal{O}_s}(x) \qquad a.e. \quad in \quad \Omega$. It follows from the previous development that:

Proposition 3.1. *If the exact solution* u *to the problem* (1)-(2) *is such that the restriction of* u *to* $\mathcal{O}_s$, *for all* $1 \le s \le S$, *lies in* $C^3(\overline{\mathcal{O}}_s)$ *then*

$$|R^{\pm}_{i+\frac{1}{4}}(h, u''')| \;\le\; C\, h^2 \qquad \forall\, i \in \{\frac{1}{2},\, 1,\, \frac{3}{2},\, 2,\, ...,\, N - \frac{1}{2},\, N\} \tag{78}$$

where C *is a nonnegative mesh-independent number. Moreover the following holds:*

$$\forall\, i \in \{\frac{1}{2},\, 1,\, \frac{3}{2},\, 2,\, ...,\, N - \frac{1}{2},\, N\} R^-_{i+\frac{1}{4}}(h, u''') \;+\; R^+_{i+\frac{1}{4}}(h, u''') \;=\; 0. \tag{79}$$

3. *Step three: Approximation of reaction term integrals*

The term $\mu(x)u(x)$ in the left-hand side of the balance equation (1) is named the reaction-term. We should look for (at least) a second order approximation of the reaction-term contribution in the integral formulation of the balance equation per control-volume (see system of equations (60)). So we should perform these integral approximations for the control-volumes K_i and $K_{i+\frac{1}{2}}$, for $i \in \{1,\, 2,\, ...,\, N\}$. Using the Rectangle-centered quadrature leads to

$$\int_{K_i} \mu(x)u(x)dx \;=\; \frac{h_i}{2}[\mu(x_i)u(x_i)] \;-\; \mathcal{E}_i(h, u'') \qquad \forall i \in \{1,\, 2,\, ...,\, N\} \tag{80}$$

with (C being a nonnegative mesh independent real number)

$$|\; \mathcal{E}_i(h, u'') \;| \le Ch^3 \qquad \forall i \in \{1,\, 2,\, ...,\, N\} \tag{81}$$

if the exact solution u to (1.1)-(1.2) lies in $C^2(\overline{\mathcal{O}}_s)$, for all $s \in \{1, 2, ..., S\}$.

Since $x_{i+\frac{1}{2}}$ is not necessarily the center-point for the interval $K_{i+\frac{1}{2}}$ the Rectangle-centered quadrature does not apply. Nevertheless a simple Rectangle quadrature still applies and leads to what follows:

$$\int_{K_i} \mu(x)u(x)dx \;=\; \frac{h_i}{2}[\mu(x_i)u(x_i)] \;-\; \mathcal{E}_i(h, u'') \qquad \forall i \in \{1,\, 2,\, ...,\, N\} \tag{82}$$

with (C being a nonnegative mesh independent real number)

$$|\; \mathcal{E}_i(h, u'') \;| \le Ch^3 \qquad \forall i \in \{1,\, 2,\, ...,\, N\} \tag{83}$$

if the exact solution u to (1)-(2) lies in $C^2(\overline{\mathcal{O}}_s)$, for all $s \in \{1, 2, ..., S\}$.

Since $x_{i+\frac{1}{2}}$ is not necessarily the center-point for the interval $K_{i+\frac{1}{2}}$ the Rectangle-centered quadrature does not apply. Nevertheless a simple Rectangle quadrature still applies and leads to what follows:

$$\int_{K_{i+\frac{1}{2}}} \mu(x)u(x)dx = \frac{h_i + h_{i+1}}{4}[\mu(x_{i+\frac{1}{2}})u(x_i + \frac{1}{2})] - \mathcal{E}_{i+\frac{1}{2}}(h, u') \quad \forall i \in \{1,\, 2,\, ..., N-1\}. \tag{84}$$

with

$$|\ \mathcal{E}_{i+\frac{1}{2}}(h, u')\ | \leq\ Ch^2 \qquad \forall i \in \{1,\, 2,\, ...,\, N-1\} \tag{85}$$

as soon as the exact solution u to (1)-(2) lies in $C^1(\overline{\mathcal{O}}_s)$, for all $s \in \{1, 2, ..., S\}$.

4. *Step four (and the Last one): Equations of the discrete mass conservation principle per control-volume*

Let us start with introducing the following simplified notations: $u_{i-\frac{1}{2}} \equiv u(x_{i-\frac{1}{2}})$, $u_i \equiv u(x_i)$, $u_{i+\frac{1}{2}} \equiv u(x_{i+\frac{1}{2}})$, $u_{i+1} \equiv u(x_{i+1})$, $\mu_i \equiv \mu(x_i)$, $\mu_{i+\frac{1}{2}} \equiv \mu(x_{i+\frac{1}{2}})$, ...

We deduce from the system of equations (60) and from the steps two and three above that in the control-volumes K_i and $K_{i+\frac{1}{2}}$ we have the following discrete mass conservation principle for all $i = 1, \cdots, N$:

$$\left\{ \lambda_i \frac{[u_i - u_{i+\frac{1}{2}}]}{h_i/2} \ +\ \lambda_i \frac{[u_i - u_{i-\frac{1}{2}}]}{h_i/2} \ +\ \frac{h_i}{2}\mu_i u_i \ =\ \frac{h_i}{2}\langle f\rangle_i \ +\ R^-_{i+\frac{1}{4}}(h, u''') \ +\ R^+_{i-\frac{1}{4}}(h, u''') \ +\ \mathcal{E}_i(h, u'') \right. \tag{86}$$

and

$$\forall i = 1, \ldots, N \ \begin{cases} \lambda_{i+1} \frac{[u_{i+\frac{1}{2}} - u_{i+1}]}{h_{i+1}/2} + \lambda_i \frac{[u_{i+\frac{1}{2}} - u_i]}{h_i/2} + \frac{[h_i + h_{i+1}]}{4}\mu_{i+\frac{1}{2}} u_{i+\frac{1}{2}} = \\ \frac{[h_i + h_{i+1}]}{4}\big\langle f\big\rangle_{i+\frac{1}{2}} + R^-_{i+\frac{1}{4}}(h, u''') + R^+_{i+\frac{1}{4}}(h, u''') + \mathcal{E}_i(h, u\,'') \end{cases} \tag{87}$$

where the following truncation errors $R^\pm_{i\pm\frac{1}{4}}(h, u''')$, $R^-_{i+\frac{3}{4}}(h, u''')$, $\mathcal{E}_i(h, u')$ and $\mathcal{E}_{i+\frac{1}{2}}(h, u')$ are introduced in what precedes with their respective estimates as well.

3.3. Definition of a New Finite Volume Scheme

From the system of equations (86-87) someone can easily see that when the truncation errors are neglected the discrete unknowns $\{u_i\}_{i=1}^{N}$ and $\{u_{i+\frac{1}{2}}\}_{i=1}^{N-1}$ satisfy the following system of "equations":

$$\lambda_i \frac{[u_i - u_{i+\frac{1}{2}}]}{h_i/2} \ +\ \lambda_i \frac{[u_i - u_{i-\frac{1}{2}}]}{h_i/2} \ +\ \frac{h_i}{2}\mu_i u_i \ \approx\ \frac{h_i}{2}\langle f\rangle_i \qquad \forall\, i = 1, \cdots, N \tag{88}$$

and

$$\lambda_{i+1} \frac{[u_{i+\frac{1}{2}} - u_{i+1}]}{h_{i+1}/2} \ +\ \lambda_i \frac{[u_{i+\frac{1}{2}} - u_i]}{h_i/2} + \frac{[h_i + h_{i+1}]}{4}\mu_{i+\frac{1}{2}} u_{i+\frac{1}{2}} \approx \frac{[h_i + h_{i+1}]}{4}\langle f\rangle_{i+\frac{1}{2}} \quad \forall\, i = 1, \cdots, N-1. \tag{89}$$

Inspired by the preceding system, let us introduce the following discrete problem:

Find in the discrete space $\mathbb{S}_0^{\mathfrak{M},0}$ *a function*

$$\overline{u}_h = \sum_{i=1}^{N} \overline{u}_i \mathbf{1}_{K_i} \ +\ \sum_{i=1}^{N-1} \overline{u}_{i+\frac{1}{2}} \mathbf{1}_{K_{i+\frac{1}{2}}} \tag{90}$$

such that its components $(\{\overline{u}_i\}_{i=1}^{N}\ ,\ \{\overline{u}_{i+\frac{1}{2}}\}_{i=1}^{N-1})$, *in the natural basis of* $\mathbb{S}_0^{\mathfrak{M},0}$, *are characterized as solution of the following system of equations*:

$$\lambda_i \frac{[\overline{u}_i - \overline{u}_{i+\frac{1}{2}}]}{h_i/2} \ +\ \lambda_i \frac{[\overline{u}_i - \overline{u}_{i-\frac{1}{2}}]}{h_i/2} \ +\ \frac{h_i}{2}\mu_i \overline{u}_i \ =\ \frac{h_i}{2}\langle f\rangle_i \qquad \forall\, i = 1, \cdots, N \tag{91}$$

and

$$\lambda_{i+1} \frac{[\overline{u}_{i+\frac{1}{2}} - \overline{u}_{i+1}]}{h_{i+1}/2} \ +\ \lambda_i \frac{[\overline{u}_{i+\frac{1}{2}} - \overline{u}_i]}{h_i/2} \ +\frac{[h_i + h_{i+1}]}{4}\mu_{i+\frac{1}{2}} \overline{u}_{i+\frac{1}{2}} \ =\ \frac{[h_i + h_{i+1}]}{4}\langle f\rangle_{i+\frac{1}{2}} \quad \forall\, i = 1 \cdots, N-1 \tag{92}$$

Definition 3.4. (*New Finite Volume Scheme*)

The system of equations (91)-(92) *is what we name a New Finite Volume Scheme for the one- dimensional Diffusion-Reaction Problem* (1)-(2) *that involves piecewise constant diffusion coefficients.*

Note that the discrete scheme (91)-(92) is not a 1-D version of the conventional Discrete Duality Finite Volumes (DDFV, for short). This is a new class of DDFV to be extended and deeply explored in higher space dimensions. Concerning the conventional DDFV that have appeared in two formulations, the first one can be seen in the pioner works of F. Herlemine and A. Njifenjou - I. Moukouop Nguena.[9, 10]. See also the following references.[11, 12, 13, 14, 15]. The second formulation of the conventional DDFV has appeared a few years later and was based on the concepts of discrete gradient, discrete divergence and discrete version of Green's formulae. The first investigator to develop this kind of ideas is P. Omnes and his collaborators.[8, 17]. Then after many developments of the second formulation of DDFV have been made in various situations (linear and non-linear diffusion operators). For learning more about it, see for instance the following works and references therein.[8, 16, 17].

4. Theoretical Analysis of the New Finite Volume Scheme

We intend to expose in the current section some important mathematical properties of the New Finite Volume Scheme. Let us start with proving existence, uniqueness and stability of $\overline{u}_h$ in $\mathbb{S}_0^{\mathfrak{M},0}$.

4.1. Existence, Uniqueness and Stability for the Solution to the System (90)-(92)

Let v_h be a discrete function from the space $\mathbb{S}_0^{\mathfrak{M},0}$, chosen arbitrarily, with components $(\{v_i\}_{i=1}^N, \{v_{j+\frac{1}{2}}\}_{j=1}^{N-1}\})$ in the basis $(\{\mathbf{1}_{K_i}\}_{i=1}^N, \{\mathbf{1}_{K_{j+\frac{1}{2}}}\}_{j=1}^{N-1}\})$. Multiplying the two sides of equations (91) and (92) with respectively v_i and $v_{j+\frac{1}{2}}$, summing on $i \in \{1, 2, ..., N\}$ and on $j \in \{1, 2, ..., N-1\}$, and doing some re-organization of terms, lead to the following variational problem:

Find $\overline{u}_h \in \mathbb{S}_0^{\mathfrak{M},0}$ such that:

$$\sum_{i=1}^{N} \frac{\lambda_i}{h_i/2}\left[\left(\overline{u}_i - \overline{u}_{i-\frac{1}{2}}\right)\left(v_i - v_{i-\frac{1}{2}}\right) + \left(\overline{u}_i - \overline{u}_{i+\frac{1}{2}}\right)\left(v_i - v_{i+\frac{1}{2}}\right)\right] + \sum_{i=1}^{N}\left[\frac{h_i}{2}\mu_i\overline{u}_i v_i + \right.$$
$$\left. + \frac{h_i + h_{i+1}}{4}\mu_{i+\frac{1}{2}}\overline{u}_{i+\frac{1}{2}}v_{i+\frac{1}{2}}\right] = \sum_{i=1}^{N}\left[\frac{h_i}{2}\langle f\rangle_i v_i + \left(\frac{h_i + h_{i+1}}{4}\right)\langle f\rangle_{i+\frac{1}{2}} v_{i+\frac{1}{2}}\right] \forall v_h \in \mathbb{S}_0^{\mathfrak{M},0}. \tag{93}$$

In terms of discrete gradient the previous variational problem can be re-written as

Find $\overline{u}_h \in \mathbb{S}_0^{\mathfrak{M},0}$ such that:

$$\sum_{i=1}^{N}\frac{\lambda_i h_i}{2}([\nabla^{\mathfrak{D}}\overline{u}_h]_{i-\frac{1}{4}}[\nabla^{\mathfrak{D}}v_h]_{i-\frac{1}{4}} + [\nabla^{\mathfrak{D}}\overline{u}_h]_{i+\frac{1}{4}}[\nabla^{\mathfrak{D}}v_h]_{i+\frac{1}{4}}) + \sum_{i=1}^{N}\left[\frac{h_i}{2}\mu_i\overline{u}_i v_i + \frac{h_i + h_{i+1}}{4}\mu_{i+\frac{1}{2}}\overline{u}_{i+\frac{1}{2}}v_{i+\frac{1}{2}}\right] =$$
$$= \sum_{i=1}^{N}\left[\frac{h_i}{2}\langle f\rangle_i v_i + \left(\frac{h_i + h_{i+1}}{4}\right)\langle f\rangle_{i+\frac{1}{2}}v_{i+\frac{1}{2}}\right] \quad \forall v_h \in \mathbb{S}_0^{\mathfrak{M},0} \quad . \tag{94}$$

First of all, remark that the following equivalence holds:

Proposition 4.1. (Equivalence between the two formulations)

Any function from the discrete functional space $\mathbb{S}_0^{\mathfrak{M},0}$ *is a solution of the system of equations* (91)-(92) *if and only if it is a solution of the variational equation* (94).

Proof Follow closely the arguments developed for a similar result in [4] (pages 23 and 24).

Proposition 4.2. (Existence, Uniqueness and Stability Results)

1. *There exists a unique function* $\overline{u}_h$ *in the space* $\mathbb{S}_0^{\mathfrak{M},0}$ *solving the variational equation* (94).
2. *Moreover the solution* $\overline{u}_h$ *to* (94) *satisfies the following inequality* (*named Stability inequality*):

$$\| \overline{u}_h \|_{\mathcal{H}_0^1(\Omega),\mathfrak{M},\mathfrak{D}} \leq C \| f \|_{L^2(\Omega)} . \tag{95}$$

where C is a mesh independent nonnegative real number.

Proof Recall that the space $\mathbb{S}_0^{\mathfrak{M},0}$ is a closed subspace of the Hilbert space $\mathbb{S}^{\mathfrak{M},0}$ which is equipped with its standard inner product $(.,.)_{\mathcal{H}^1(\Omega),\mathfrak{M},\mathfrak{D}}$. So $\mathbb{S}_0^{\mathfrak{M},0}$ is also a Hilbert space with respect to this inner product. That being said, let us set:

$$\mathfrak{B}(w_h, v_h) \stackrel{def}{=} \sum_{i=1}^{N} \frac{\lambda_i h_i}{2}\left([\nabla^{\mathfrak{D}} w_h]_{i-\frac{1}{4}}[\nabla^{\mathfrak{D}} v_h]_{i-\frac{1}{4}}[\nabla^{\mathfrak{D}} w_h]_{i+\frac{1}{4}}[\nabla^{\mathfrak{D}} v_h]_{i+\frac{1}{4}}\right)+ + \sum_{i=1}^{N}\left[\frac{h_i}{2}\mu_i w_i v_i + \frac{h_i+h_{i+1}}{4}\mu_{i+\frac{1}{2}} w_{i+\frac{1}{2}} v_{i+\frac{1}{2}}\right] \forall w_h, v_h \in \mathbb{S}_0^{\mathfrak{M},0} \tag{96}$$

and

$$L(v_h) = \sum_{i=1}^{N}\left[\frac{h_i}{2}\langle f\rangle_i v_i + \left(\frac{h_i+h_{i+1}}{4}\right)\langle f\rangle_{i+\frac{1}{2}} v_{i+\frac{1}{2}}\right] \quad \forall v_h \in \mathbb{S}_0^{\mathfrak{M},0}. \tag{97}$$

1. Let us prove *Existence* and *Uniqueness* of the solution to the variational equation (94) by an application of the Lax-Milgram theorem. It is obviously seen that $\mathfrak{B}(.,.)$ and $L(.)$ are respectively a bilinear form and a linear form over $\mathbb{S}_0^{\mathfrak{M},0}$. Let us check if the conditions of Lax-Milgram are satisfied by these two forms (see for instance $[5]$ to learn more about theoretical aspects of Lax-Milgram theorem, notably the proof of this theorem; see for instance $[4, 6]$ for the application of Lax-Milgram theorem to Numerical Analysis of discrete models for diffusion problems).

a. *Continuity* of $\mathfrak{B}(.,.)$: From the fact that $(A+B)^2 \leq 2[A^2+B^2]$ for all real numbers A and B, we easily get that:

$$\mid \mathfrak{B}(w_h, v_h) \mid \leq 2\left[\sum_{i=1}^{N}\frac{\lambda_i h_i}{2}\left(\mid [\nabla^{\mathfrak{D}} w_h]_{i-\frac{1}{4}} \mid + \mid [\nabla^{\mathfrak{D}} w_h]_{i+\frac{1}{4}} \mid\right)\left(\mid [\nabla^{\mathfrak{D}} v_h]_{i-\frac{1}{4}} \mid + \mid [\nabla^{\mathfrak{D}} v_h]_{i+\frac{1}{4}} \mid\right)\right]^2 + +2\left[\int_{\Omega}\mu(x) \mid w_h(x) \mid\mid v_h(x) \mid dx\right]^2 \quad \forall w_h, v_h \in \mathbb{S}_0^{\mathfrak{M},0} \tag{98}$$

Thanks to assumptions in (3) and (5) we can get what follows from the previous inequality

$$\mid \mathfrak{B}(w_h, v_h) \mid \leq 2\lambda_+^2\left[\sum_{i=1}^{N}\frac{h_i}{2}\left(\mid [\nabla^{\mathfrak{D}} w_h]_{i-\frac{1}{4}} \mid + \mid [\nabla^{\mathfrak{D}} w_h]_{i+\frac{1}{4}} \mid\right)\left(\mid [\nabla^{\mathfrak{D}} v_h]_{i-\frac{1}{4}} \mid + \mid [\nabla^{\mathfrak{D}} v_h]_{i+\frac{1}{4}} \mid\right)\right]^2 + +2 \parallel \mu \parallel_{L^2(\Omega)}^2 \left[\int_{\Omega} \mid w_h(x) \mid\mid v_h(x) \mid dx\right]^2 \forall w_h, v_h \in \mathbb{S}_0^{\mathfrak{M},0}. \tag{99}$$

From a double application of Cauchy-Schwartz inequality in the right-hand side of the previous inequality and thanks to Proposition 2.4 (discrete version of Poincaré-Friedrichs) we obtain the continuity of the bilinear form $\mathfrak{B}(.,.)$.

b. *Coercivity* of $\mathfrak{B}(.,.)$: Let w_h be an arbitrarily chosen function from the space $\mathbb{S}_0^{\mathfrak{M},0}$. So we have

$$\mathfrak{B}(w_h, w_h) = \sum_{i=1}^{N}\frac{\lambda_i h_i}{2}\left(([\nabla^{\mathfrak{D}} w_h]_{i-\frac{1}{4}})^2 + ([\nabla^{\mathfrak{D}} w_h]_{i+\frac{1}{4}})^2\right) + \sum_{i=1}^{N}\left(\frac{h_i\mu_i}{2}w_i^2 + \frac{h_i+h_{i+1}}{4}\mu_{i+\frac{1}{2}}w_{i+\frac{1}{2}}^2)\right) \quad \geq$$

$$\geq \quad \lambda_- \sum_{i=1}^{N}\frac{h_i}{2}\left(([\nabla^{\mathfrak{D}} w_h]_{i-\frac{1}{4}})^2 + ([\nabla^{\mathfrak{D}} w_h]_{i+\frac{1}{4}})^2\right) = \lambda_- \parallel \nabla^{\mathfrak{D}} w_h \parallel_{L^2(\Omega),\mathfrak{D}}^2 . \tag{100}$$

where λ_- is a mesh independent nonnegative number coming from the assumption (3).

c. *Continuity* of the linear form $L(.)$: Let w_h be an arbitrarily chosen function from the space $\mathbb{S}_0^{\mathfrak{M},0}$. So we have to show that there exists a mesh independent nonnegative real number β such that

$$\mid L(w_h) \mid \leq \beta \parallel \nabla^{\mathfrak{D}} w_h \parallel_{L^2(\Omega),\mathfrak{D}} \qquad \forall w_h \in \mathbb{S}_0^{\mathfrak{M},0}. \tag{101}$$

There exist (at least) three ways to prove the inequality (101).

1) *First way*: Remark that $L(.)$ is a *linear* map defined on a *finite dimensional* space. So $L(.)$ is necessarily continuous.

2) *Second way*: Remark that

$$L(w_h) = \sum_{i=1}^{N}\int_{K_i} f(x)w_h(x)dx + \sum_{i=1}^{N}\int_{K_{i+\frac{1}{2}}} f(x)w_h(x)dx \int_{\Omega} f(x)w_h(x)dx \tag{102}$$

Since the norm $\| \,.\, \|_{L^2(\Omega),\mathfrak{M}}$ is the restriction to $\mathbb{S}^{\mathfrak{M},0}$ of the standard norm of the well-known Hilbert space $L^2(\Omega)$, the Cauchy-Schawrz inequality applies and we get

$$| L(w_h) | \;\le\; \| f \|_{L^2(\Omega)} \| w_h \|_{L^2(\Omega),\mathfrak{M}} \qquad \forall w_h \in \mathbb{S}_0^{\mathfrak{M},0}. \tag{103}$$

It follows from the discrete version of Poincaré-Friedrichs inequality that there exists a mesh independent nonnegative real number C such that

$$| L(w_h) | \;\le\; C \| \nabla^{\mathfrak{D}} w_h \|_{L^2(\Omega),\mathfrak{D}} \qquad \forall w_h \in \mathbb{S}_0^{\mathfrak{M},0}. \tag{104}$$

The continuity of $L(.)$ is proven.

3) *Third way*: In virtue of the identity $(A+B)^2 \le 2A^2 + 2B^2$ that holds for all $A, B \in \mathbb{R}$, we have what follows for all $w \in \mathbb{S}_0^{\mathfrak{M},0}$:

$$| L(v_h) | \le 2\left[\sum_{i=1}^{N} w_i \int_{K_i} f(x)dx\right]^2 + 2\left[\sum_{i=1}^{N} w_{i+\frac{1}{2}} \int_{K_{i+\frac{1}{2}}} f(x)dx\right]^2$$

(from discrete Cauchy-Schwarz inequality we get)

$$\le 2\left[\sum_{i=1}^{N} \frac{h_i}{2} w_i^2\right]\left[\sum_{i=1}^{N} \frac{1}{h_i/2}\left(\int_{K_i} f(x)dx\right)^2\right] + 2\left[\sum_{i=1}^{N} \frac{h_i+h_{i+1}}{4} w_{i+\frac{1}{2}}^2\right]\left[\sum_{i=1}^{N} \frac{4}{h_i+h_{i+1}}\left(\int_{K_{i+\frac{1}{2}}} f(x)dx\right)^2\right]$$

(Thanks to Cauchy-Schwarz's inequality we have)

$$\le 2\left[\sum_{i=1}^{N} \frac{h_i}{2} w_i^2\right]\left[\sum_{i=1}^{N} \int_{K_i} [f(x)]^2 dx\right] + 2\left[\sum_{i=1}^{N} \frac{h_i+h_{i+1}}{4} w_{i+\frac{1}{2}}^2\right]\left[\int_{K_{i+\frac{1}{2}}} [f(x)]^2 dx\right]$$

Therefore

$$| L(w_h) |^2 \le 2 \| f \|_{L^2(\Omega)}^2 \| w_h \|_{L^2(\Omega),\mathfrak{M}}^2 . \tag{105}$$

Thanks to the discrete version of Poincaré-Friedrichs (see Proposition 2.4, relation (46) above) we could conclude that $L(.)$ is a continuous linear form.

2. The *Stability* of the discrete solution $\overline{u}_h$ straightly follows from the coercivity of the bilinear form $\mathfrak{B}(.,.)$ and the continuity of the linear form $L(.)$.

4.2. Matrix Properties of the Scheme (91)-(92)

The matrix form of the new Finite Volume Scheme (91)-(92) may be expressed as follows:

$$\begin{pmatrix} A_h & B_h \\ (B_h)^t & C_h \end{pmatrix}\begin{pmatrix} \overline{U}_{cc}^h \\ \overline{U}_{vc}^h \end{pmatrix} = \begin{pmatrix} F_{cc}^h \\ F_{vc}^h \end{pmatrix} \tag{106}$$

where we have set :

$$\overline{U}_{cc}^h = \{\overline{u}_i\}_{1\le i\le N} \qquad \text{and} \qquad \overline{U}_{vc}^h = \left\{\overline{u}_{i+\frac{1}{2}}\right\}_{1\le i\le N-1} \tag{107}$$

and where:

F_{cc}^h is a sub-vector with N components defined only by the right hand side of (91) and F_{vc}^h a sub-vector with $(N-1)$ components defined only by the right hand side of (92) as the boundary conditions are of homogeneous Dirichlet ones. Concerning the sub-matrices A_h, C_h and B_h, note that the first two ones are respectively $N \times N$ and $(N-1)\times(N-1)$ diagonal matrices while B_h is a $N \times (N-1)$ matrix with a maximum of two coefficients different from 0 per line. At last $(\cdot)^t$ is the matrix transposition operator. So the symmetry and the sparse structure of the matrix associated with the new finite volume scheme are established.

Proposition 4.3. *The matrix M_h associated with the finite volume scheme* (91)-(92) for solving (1)-(2) *is symmetric, positive definite and monotone.*

Proof Since the symmetry of M_h is obvious let us concentrate on positive definiteness and monotonicity.

1. *Positive definiteness:* It follows straightly from the coercivity of the bilinear form $\mathfrak{B}(.,.)$ introduced in the proof of Proposition 4.2.

2. *Monotonicity:* By definition M_h is monotone if the components of any solution $\overline{u}_h$ to the system (91)-(92) are positive as soon as the components of $\Pi_{\mathfrak{M}} f$ are positive (see Proposition 2.5 for the definition of the operator $\Pi_{\mathfrak{M}}$). Recall that the following characterization of monotone matrix holds: M_h is monotone if and only if M_h is nonsingular and the coefficients of its inverse are positive (see for instance [25]. At least there are two ways to prove that M_h is monotone.

a. *The first way is a classical technique*: We start with setting

$$\overline{u}_{min} = \min\{\overline{u}_{\frac{1}{2}}, \overline{u}_1, \overline{u}_{\frac{3}{2}}, ..., \overline{u}_{i-\frac{1}{2}}, \overline{u}_i, \overline{u}_{i+\frac{1}{2}}, ..., \overline{u}_{N-\frac{1}{2}}, \overline{u}_N\}$$

and

$$\overline{r} = \min\{r \in \{\frac{1}{2}, 1, \frac{3}{2}, ..., i-\frac{1}{2}, i, i+\frac{1}{2}, ..., N-\frac{1}{2}, N\}$$

$$\overline{u}_r = \overline{u}_{min}\}.$$

Let us suppose that

$$\frac{1}{2} < \overline{r} \leq N. \tag{108}$$

From the previous assumption it is easily seen that there are two possibilities:

First possibility: $\overline{r} \in \{1, 2, ..., i-1, i, i+1, ..., N-1, N\}$. In this case the following inequalities hold:

$$0 \leq \lambda_{\overline{r}} \frac{[\overline{u}_{\overline{r}} - \overline{u}_{\overline{r}+\frac{1}{2}}]}{h_{\overline{r}}/2} + \lambda_{\overline{r}} \frac{[\overline{u}_{\overline{r}} - \overline{u}_{\overline{r}-\frac{1}{2}}]}{h_{\overline{r}}/2} + \frac{h_{\overline{r}}}{2} \mu_{\overline{r}} \overline{u}_{\overline{r}} < 0. \tag{109}$$

This is absurd. So $\overline{r} \in \{1, 2, ..., i-1, i, i+1, ..., N-1, N\}$ is not possible.

Second possibility: $\overline{r} \in \{\frac{3}{2}, ..., i-\frac{1}{2}, i+\frac{1}{2}, ..., N-\frac{1}{2}\}$, with $\widehat{i} < \overline{r} < \widehat{i}+1$, where $\widehat{i}$ is an integer from the set $\{1, 2, ..., N-1\}$. In this case the following inequalities hold:

$$0 \leq \lambda_{\widehat{i}+1} \frac{[\overline{u}_{\overline{r}} - \overline{u}_{\widehat{i}+1}]}{h_{\widehat{i}+1}/2} + \lambda_{\widehat{i}} \frac{[\overline{u}_{\overline{r}} - \overline{u}_{\widehat{i}}]}{h_{\widehat{i}}/2} + \frac{[h_{\widehat{i}} + h_{\widehat{i}+1}]}{4} \mu_{\overline{r}} \overline{u}_{\overline{r}} < 0. \tag{110}$$

This is also absurd. So $\overline{r} \in \{\frac{3}{2}, ..., i-\frac{1}{2}, i+\frac{1}{2}, ..., N-\frac{1}{2}\}$ is not a possibility. We conclude that the assumption (108) is wrong. In consequence $\overline{r} = \frac{1}{2}$. This implies that all the components of $\overline{u}_h$ are positive since $\overline{u}_{\frac{1}{2}} = 0$.

b. *The second way is a geometric technique:* According to our knowledge, it has been exposed for the first time in a work from A. Njifenjou for a two-dimensional diffusion problem.[7]. We are going right now to apply it to the new finite volume scheme (91)-(92).

Let us suppose that the right-hand side of the system (91)-(92) is positive. We should deduce that all the components $\Big(\overline{u}_i]_{i=1}^{N}; \overline{u}_{i+\frac{1}{2}}]_{i=1}^{N-1}\Big)$ of its solution $\overline{u}_h$ are positive.

Let us denote by $\overline{u}_\sigma$ the smallest of the quantities $\overline{u}_j$, with $j \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, ..., N-\frac{1}{2}, N\}$. There are two possibilities: σ is either an integer or a non-integer rational number. Let us analyze the case where σ is an integer number. The left-hand side of the discrete balance equation in the control-volume K_σ satisfies what follows:

$$0 \leq \lambda_\sigma \frac{[\overline{u}_\sigma - \overline{u}_{\sigma+\frac{1}{2}}]}{h_{\overline{r}}/2} + \lambda_\sigma \frac{[\overline{u}_\sigma - \overline{u}_{\sigma-\frac{1}{2}}]}{h_\sigma/2} + \frac{h_\sigma}{2} \mu_\sigma \overline{u}_\sigma \leq 0. \tag{111}$$

So we have

$$\overline{u}_{\sigma-\frac{1}{2}} = \overline{u}_\sigma = \overline{u}_{\sigma+\frac{1}{2}} \tag{112}$$

If $\sigma = 1$ or $\sigma = N$ the proof is ended. Otherwise, consider the straight semi-line Δ_σ with origin the node x_σ and passing through $x_{\frac{1}{2}}$. Writing the discrete balance for the node $x_{\sigma-\frac{1}{2}}$ and accounting with (112) we see that $\overline{u}_\sigma = \overline{u}_{\sigma-1} = \overline{u}_{[\sigma-1]-\frac{1}{2}}$. If $[\sigma-1] = 1$ the proof is ended, otherwise repeat the procedure until the equality $[\sigma-1] = 1$ holds. This should happen after a finite number of iterations. The case where σ is a non-integer rational number is analyzed following the same way as the case where σ is an integer number.

4.3. Error Estimates in $\|\cdot\|_{L^2(\Omega),\mathfrak{D}}$, $\|\cdot\|_{L^2(\Omega),\mathfrak{M}}$ and $\|\cdot\|_{L^\infty(\Omega),\mathfrak{M}}$

Let us set

$$\begin{cases} e_i \overset{def}{=} u(x_i) - \overline{u}_i & \forall\, 1 \leq i \leq N \\ e_{i+\frac{1}{2}} \overset{def}{=} u(x_{i+\frac{1}{2}}) - \overline{u}_{i+\frac{1}{2}} & \forall\, 0 \leq i \leq N \end{cases}$$

Recall that $u(a) \equiv u(x_{\frac{1}{2}}) \overset{def}{=} \overline{u}_{\frac{1}{2}}$ and that $u(b) \equiv u(x_{N+\frac{1}{2}}) \overset{def}{=} \overline{u}_{N+\frac{1}{2}}$. In consequence we have the following discrete boundary conditions:

$$e_{\frac{1}{2}} = e_{N+\frac{1}{2}} = 0. \tag{113}$$

We can now define the error function e_h as follows

$$e_h(x) = \sum_{i=1}^{N} e_i \mathbf{1}_{K_i}(x) + \sum_{i=1}^{N-1} e_{i+\frac{1}{2}} \mathbf{1}_{K_{i+\frac{1}{2}}}(x) \ a.e. \ in\ \Omega.$$

It is then clear that e_h lies in $\mathbb{S}_0^{\mathfrak{M},0}$. It is easily seen that this discrete function is solution of a system of the same type as (86)-(87). Indeed, subtracting side by side equation (91) from equation (86) and equation (92) from (87) lead to the so-called Error function system that reads as

$$\begin{cases} \frac{2\lambda_i}{h_i/2}\left(e_i - e_{i+\frac{1}{2}}\right) + \frac{2\lambda_i}{h_i/2}\left(e_i - e_{i-\frac{1}{2}}\right) + \frac{h_i}{2}\mu_i e_i = R^-_{i+\frac{1}{4}} + R^+_{i-\frac{1}{4}} + \mathcal{E}_i \ \forall i = 1,\dots,N \\ \frac{2\lambda_{i+1}}{h_{i+1/2}}\left(e_{i+\frac{1}{2}} - e_{i+1}\right) + \frac{2\lambda_i}{h_i/2}\left(e_{i+\frac{1}{2}} - e_i\right) + \frac{h_i+h_{i+1}}{4}\mu_{i+\frac{1}{2}} e_{i+\frac{1}{2}} = R^-_{i+\frac{3}{4}} + R^+_{i+\frac{1}{4}} + \mathcal{E}_{i+\frac{1}{2}} \ \forall i = 1,\dots,N-1 \end{cases} \tag{114}$$

Let us now investigate some estimates of the Error function with respect to the norm $\|\ .\ \|_{\mathcal{H}^1_0(\Omega),\mathfrak{M}}$ also denoted by $\|\ .\ \|_{L^2(\Omega),\mathfrak{D}}$ and the standard norm of $L^2(\Omega)$. For that purpose let us multiply the two sides of the first equation of the system (114) by e_i and then we sum on $i \in \{1, 2, ..., N\}$. Let us repeat the same operations with the second equation of the same system, but with $e_{i+\frac{1}{2}}$ instead of e_i, and $i \in \{1, 2, ..., N-1\}$ instead of $i \in \{1, 2, ..., N\}$. Adding side by side the two previous sums and re-ordering the terms, accounting with (79), lead to what follows:

$$\mathfrak{B}(e_h, e_h) = \sum_{i=1}^{N} R^-_{i+\frac{1}{4}}\left(e_i - e_{i+\frac{1}{2}}\right) + \sum_{i=1}^{N} R^+_{i-\frac{1}{4}}\left(e_i - e_{i-\frac{1}{2}}\right) + \sum_{i=1}^{N} \mathcal{E}_i e_i + \sum_{i=1}^{N-1} \mathcal{E}_{i+\frac{1}{2}} e_{i+\frac{1}{2}}. \tag{115}$$

where $\mathfrak{B}(.,.)$ is defined by (96) in Subsection 4.1 . From the discrete version of Cauchy-Schwarz inequality we get

$$|\ \mathfrak{B}(e_h, e_h)\ |^2 \le 4\left[\sum_{i=1}^{N} R^-_{i+\frac{1}{4}}\left(e_i - e_{i+\frac{1}{2}}\right)\right]^2 + \left[\sum_{i=1}^{N} R^+_{i-\frac{1}{4}}\left(e_i - e_{i-\frac{1}{2}}\right)\right]^2 + 4\left[\sum_{i=1}^{N} \mathcal{E}_i e_i\right]^2 + 4\left[\sum_{i=1}^{N} \mathcal{E}_{i+\frac{1}{2}} e_{i+\frac{1}{2}}\right]^2. \tag{116}$$

Applying again the discrete version of Cauchy-Schwarz to the square of the first two sums from the right-hand side of the previous inequality yields, accounting with (78) from Proposition 3.1 (note that in what follows C represents diverse mesh independent nonnegative numbers):

$$4\left[\sum_{i=1}^{N} R^-_{i+\frac{1}{4}}(e_i - e_{i+\frac{1}{2}})\right]^2 \le C\,h^4 \sum_{i=1}^{N} \frac{1}{h_i/2}(e_i - e_{i+\frac{1}{2}})^2 \tag{117}$$

and

$$4\left[\sum_{i=1}^{N} R^+_{i-\frac{1}{4}}(e_i - e_{i-\frac{1}{2}})\right]^2 \le C\,h^4 \sum_{i=1}^{N} \frac{1}{h_i/2}(e_i - e_{i-\frac{1}{2}})^2. \tag{118}$$

Repeating the same exercise with the square of the two last sums from the right-hand side of (116) and accounting with (20), (83) and (85), leads to

$$4\left[\sum_{i=1}^{N} \mathcal{E}_i e_i\right]^2 \le C\,h^4 \sum_{i=1}^{N} h_i e_i^2 = C\,h^4 \parallel \overline{\mathbb{P}}_{\mathfrak{M}}(e_h) \parallel^2_{L^2(\Omega),\mathfrak{M}} \tag{119}$$

and

$$4\left[\sum_{i=1}^{N-1} \mathcal{E}_{i+\frac{1}{2}} e_{i+\frac{1}{2}}\right]^2 \le C\,h^2 \sum_{i=1}^{N-1} \frac{h_i + h_{i+1}}{4} e^2_{i+\frac{1}{2}} = C\,h^2 \parallel \underline{\mathbb{P}}_{\mathfrak{M}}(e_h) \parallel^2_{L^2(\Omega),\mathfrak{M}} \tag{120}$$

Applying the Lemma 2.1 to the terms

$$C\,h^4 \parallel \overline{\mathbb{P}}_{\mathfrak{M}}(e_h) \parallel^2_{L^2(\Omega),\mathfrak{M}} \ \ and\, C\,h^2 \parallel \underline{\mathbb{P}}_{\mathfrak{M}}(e_h) \parallel^2_{L^2(\Omega),\mathfrak{M}}$$

leads to

$$4\left[\sum_{i=1}^{N} \mathcal{E}_i e_i\right]^2 \le C\,h^4 \parallel \nabla^{\mathfrak{D}}(e_h) \parallel^2_{L^2(\Omega),\mathfrak{D}} \tag{121}$$

and

$$4\left[\sum_{i=1}^{N-1} \mathcal{E}_{i+\frac{1}{2}} e_{i+\frac{1}{2}}\right]^2 \le C\,h^2 \parallel \nabla^{\mathfrak{D}}(e_h) \parallel^2_{L^2(\Omega),\mathfrak{D}} \tag{122}$$

On the other hand, it follows from inequalities (117)-(118) that

$$4\left[\sum_{i=1}^{N} R^{-}_{i+\frac{1}{4}}(e_i - e_{i+\frac{1}{2}})\right]^2 + 4\left[\sum_{i=1}^{N} R^{+}_{i-\frac{1}{4}}(e_i - e_{i-\frac{1}{2}})\right]^2 \le C\,h^4 \parallel \nabla^{\mathfrak{D}}(e_h) \parallel^2_{L^2(\Omega),\mathfrak{D}} . \tag{123}$$

From the inequalities (116), (121), (122), (123) and thanks to the coercivity of the bilinear form $\mathfrak{B}(.,.)$ we can state what follows:

Theorem 4.1. (*Error Estimates*)

Recall that the finite family of non empty subintervals $\{\mathcal{O}_s\}_{s\in S}$ (*associated with the diffusion coefficient* $\lambda(.)$) *defines a partition of* Ω. *Consider the assumptions* (3), (5), (20) *and the following ones:*

(i) *The discontinuity points of the diffusion coefficient* $\lambda(.)$ *are part of the set* $\{x_{i+\frac{1}{2}}\}_{i=0}^{N}$ *associated with the primary (relatively coarse) mesh,*

(ii) *The exact solution* u *is such that the restriction* $u\,|_{\mathcal{O}_s}$ *of* u *to* $\mathcal{O}_s$ *honors the following condition:*

$$u\,|_{\mathcal{O}_s} \in C^3(\overline{\mathcal{O}}_s) \qquad \forall\, s \in S.$$

Then the finite volume approximation $\overline{u}_h$ *of the exact solution* u *to* (1)-(2) *is such that:*

a. *In the context of non uniform primary mesh elements* $\Omega_i]_{i=1}^{N}$ *combined with non negligible reaction effects, the Error function* $e_h = u_h - \overline{u}_h$ *satisfies the following estimates (first order convergence):*

$$\parallel \nabla^{\mathfrak{D}} e_h \parallel_{L^2(\Omega),\mathfrak{D}} \le C\,h, \qquad \parallel e_h \parallel_{L^2(\Omega),\mathfrak{M}} \le C\,h$$

and

$$\parallel e_h \parallel_{L^\infty(\Omega),\mathfrak{M}} \le C\,h;$$

b. *In the context of uniform primary mesh elements* $\Omega_i]_{i=1}^{N}$ *combined with non negligible reaction effects, the Error function* e_h *meets what follows (second order convergence):*

$$\parallel \nabla^{\mathfrak{D}} e_h \parallel_{L^2(\Omega),\mathfrak{D}} \le C\,h^2, \qquad \parallel e_h \parallel_{L^2(\Omega),\mathfrak{M}} \le C\,h^2$$

and

$$\parallel e_h \parallel_{L^\infty(\Omega),\mathfrak{M}} \le C\,h^2;$$

c. *In the context of pure diffusion problems (i.e. reaction effects negligible), over non-uniform primary meshes, the Error function* e_h *satisfies the following estimates (second order convergence):*

$$\parallel \nabla^{\mathfrak{D}} e_h \parallel_{L^2(\Omega),\mathfrak{D}} \le C\,h^2, \quad \parallel e_h \parallel_{L^2(\Omega),\mathfrak{M}} \le C\,h^2$$

and

$$\parallel e_h \parallel_{L^\infty(\Omega),\mathfrak{M}} \le C\,h^2.$$

Concluding remark: Note that the previous theorem asserts that one can get a second order convergence from the new finite volume scheme (91)-(92) applied to any 1-D diffusion problem involving discontinuous coefficients (under reasonable assumptions for the rest of data).

5. Conclusion

We have exposed in this work some new ideas for improving the Finite Volume approximation of fluxes in the context of 1-D flow problems governed by discontinuous diffusion coefficients. By so doing a second order convergence in adequate discrete energy norms is obtained even if the diffusion coefficients is a piece-wise constant function. There are three important features to underline concerning the proposed Finite Volume scheme:

1. The first one is the large family of freedom degrees associated with the approximate solution. We take this opportunity to put in place a polynomial reconstruction of the solution over diamond elements designed with smaller characteristic size.
2. The extreme flexibility of the proposed Finite Volume scheme is the second important feature to be underlined. Indeed, as being seen in ongoing works the proposed methods displays strong ability to easily extend to 2-D and 3-D nonlinear anisotropic diffusion problems, in general grids, following the spirit of the conventional Discrete Duality Finite Volumes.
3. The remarkable feature above all is its capability in higher space-dimension problems to avoid the computation of equivalent effective diffusion matrix to allocate to each diamond mesh element as required to the conventional Discrete Duality Finite Volumes.

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Conflicts of Interest

The authors declare no conflicts of interest.

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